An Elementary Approach to Elementary Topos Theory

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October 26, 2019

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- Standard approach: forbiddingly technical (monadicity criteria, Beck-Chevalley conditions, ...) for those who grew up on naive set theory.
- Tierney's approach: constructions are more natively "set-theoretical".

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- Standard approach to deduce existence of colimits:
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- Construction of coproducts: $X + Y$ is an equalizer:

$$X{+}Y \rightarrowtail P(PX{\times}PY) \underset{P\langle P(P\pi_{PX}\circ u_X), P(P\pi_{PY}\circ u_Y)\rangle}{\overset{u_{P(PX\times PY)}}{\rightrightarrows}} PPP(PX{\times}PY)$$

$$u : 1_E \to PP \text{ is unit} \qquad u_X(x) = \{A : PX | x \in A\}$$

Notation and Preliminaries

- Power-object definition of topos: finite limits, universal relations $\ni_X \hookrightarrow PX \times X$.

$$\frac{R \hookrightarrow X \times Y}{X \underset{\chi_R}{\rightarrow} PY}$$

$$\begin{array}{ccc} R & \longrightarrow & \ni_Y \\ {\scriptstyle i}\downarrow & \lrcorner & \downarrow \\ X \times Y & \xrightarrow{\chi_i \times 1_Y} & PY \times Y \end{array}$$

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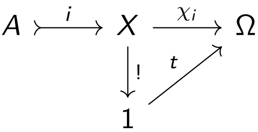
- $\mathrm{sing}_X : X \to PX$ classifies $\delta_X : X \to X \times X$.

Notation and Preliminaries

- $\exists_1 = 1 \to P1 \times 1$, aka $t : 1 \to \Omega$.

- All monos are regular:



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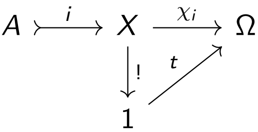

- Epi-mono factorizations are unique when they exist.

- Toposes are balanced.

Cartesian closure

- Exponentials PZ^Y exist, namely $P(Y \times Z) \cong (PZ)^Y$:

$$\frac{\dfrac{\dfrac{X \to P(Y \times Z)}{R \to X \times Y \times Z}}{X \times Y \to PZ}}{X \to PZ^Y}$$

-

$$\begin{array}{ccc} X & \longrightarrow & 1 \\ {\scriptstyle \mathrm{sing}_X} \downarrow \lrcorner & & \downarrow {\scriptstyle t} \\ PX & \xrightarrow{\tau} & P1 \end{array} \qquad \begin{array}{ccc} X^Y & \longrightarrow & 1^Y \\ \downarrow \lrcorner & & \downarrow {\scriptstyle t^Y} \\ PX^Y & \xrightarrow{\tau^Y} & P1^Y \end{array}$$

Slice theorem

- If E is a topos, then for any object X, the category E/X is also a topos. The change of base $X^* : E \to E/X$ is logical and has left and right adjoints.

- $f^* : E/Y \to (E/Y)/f \simeq E/X$, for $f : X \to Y$, is logical.

Slice theorem

- If E is a topos, then for any object X, the category E/X is also a topos. The change of base $X^* : E \to E/X$ is logical and has left and right adjoints.
- $f^* : E/Y \to (E/Y)/f \simeq E/X$, for $f : X \to Y$, is logical.
- Colimits in E/Y, when they exist, are stable under pullback $f^* : E/Y \to E/X$.

Internal logic

$$\frac{1 \times 1 \xrightarrow{t \times t} \Omega \times \Omega}{\wedge = \chi_{t \times t} : \Omega \times \Omega \to \Omega}$$

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$$\frac{[\leq] \hookrightarrow \Omega \times \Omega}{\Rightarrow = \chi_{[\leq]} : \Omega \times \Omega \to \Omega}$$

Internal logic

$$\frac{\dfrac{X \xrightarrow{!} 1 \xrightarrow{t} \Omega}{t_X : 1 \to \Omega^X = PX}}{\forall_X = \chi_{t_X} : PX \to \Omega}$$

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Define $\bigcap_X : PPX \to PX$ by

$$\bigcap \mathcal{F} = \{x : X \mid \forall_{A:PX}\, A \in_{PX} \mathcal{F} \Rightarrow x \in_X A\}$$

Construction of coproducts

- Initial object: define $0 \hookrightarrow 1$ to be "intersection all subobjects of 1", classified by

$$1 \xrightarrow{t_{P1}} PP1 \xrightarrow{\bigcap} P1$$

- **Lemma:** 0 is initial.

- Uniqueness: if $f, g : 0 \rightrightarrows X$, then $\mathrm{Eq}(f, g) \rightarrowtail 0$ is an equality, by minimality of 0 in Sub(1).
- Existence: consider

$$\begin{array}{ccccc} P & & \longrightarrow & & X \\ \downarrow & \lrcorner & & & \downarrow \mathsf{sing}_X \\ 0 & \longrightarrow & 1 & \xrightarrow{t_X} & PX \end{array}$$

Coproducts

- 0 is strict by cartesian closure, so $0 \to X$ is monic.

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- Given X, Y, disjointly embed them into $PX \times PY$:

$$X \times 1 \xrightarrow{\chi_\delta \times \chi_0} PX \times PY \qquad 1 \times Y \xrightarrow{\chi_0 \times \chi_\delta} PX \times PY$$

  $X \sqcup Y$ is the "disjoint union": the intersection of the definable family of subobjects of $PX \times PY$ containing these embeddings.

Coproducts

- **Lemma:** Any two disjoint unions of X, Y are isomorphic.
- **Proof:** If $Z = X \cup Y$ via $i : X \to Z$ and $j : Y \to Z$, then map Z into $PX \times PY$ via

$$\frac{X \overset{\langle 1_X, i\rangle}{\hookrightarrow} X \times Z \quad Y \overset{\langle 1_Y, j\rangle}{\hookrightarrow} Y \times Z}{Z \to PX \qquad Z \to PY}$$

Then $Z \to PX \times PY$ is monic. Both Z and $X \sqcup Y$ are least upper bounds of X and Y in $\mathrm{Sub}(PX \times PY)$. $\square$

Coproducts

- **Theorem:** $X \sqcup Y$ is the coproduct.

- **Proof:** Given $f : X \to B$ and $g : Y \to B$, form

$$X \overset{\langle 1_X, f\rangle}{\hookrightarrow} X \times B, \qquad Y \overset{\langle 1_Y, g\rangle}{\hookrightarrow} Y \times B.$$

Then $(X \sqcup Y) \times B \cong (X \times B) \sqcup (Y \times B)$. So both X, Y embed disjointly in $(X \sqcup Y) \times B$. Obtain

$$X \sqcup Y \hookrightarrow (X \sqcup Y) \times B. \qquad \square$$

Image factorization

- For $f : X \to Y$, define $\mathrm{im}(f)$ to be the intersection of the (definable) family of subobjects through which f factors.

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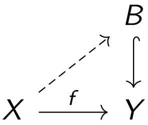
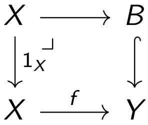
- 



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- $\mathrm{im}(f) = \bigcap_Y \{B : PY \mid f^*B = X\}$

Image factorization

- **Lemma:** $f : X \to Y$ indeed factors through $\mathrm{im}(f) : I \to Y$.

- **Proof:** We must show $f^*(\mathrm{im}(f)) = X$. But

$$
\begin{aligned}
f^*\left(\bigcap_{B \mid f^*B=X} B\right) &= \bigcap_{B \mid f^*B=X} f^*B \quad [E/Y \xrightarrow{f^*} E/X \text{ is logical}] \\
&= \bigcap_{B \mid f^*B=X} X \\
&= X \qquad\qquad \square
\end{aligned}
$$

Image factorization

- **Lemma:** $X \to \mathrm{im}(f) \hookrightarrow Y$ is the epi-mono factorization of $f : X \to Y$.

Proof: Put $I = \mathrm{im}(f)$; suppose $X \to I$ equalizes $g, h : I \rightrightarrows Z$. Then

$$X \to \mathrm{Eq}(g, h) \rightarrowtail I \hookrightarrow Y$$

makes $\mathrm{Eq}(g, h)$ a subobject through which f factors. Hence $\mathrm{Eq}(g, h) = I$ and $g = h$. $\square$

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- **Theorem:** $Y \to Q$ is the coequalizer of f, g.