A quantalic perspective on persistent and magnitude homology

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Outline

- Basics of persistent & magnitude homology
- Graphs & categories enriched over quantales
  - $(\mathbb{R}, +_p)$
- A nerve functor for each quantale
  - Vietoris-Rips complex as $N_{(\mathbb{R},\max)}$ and magnitude complex as $N_{(\mathbb{R},+)}$
- Localization along $\mathbb{R}$
- A natural transformation from persistent to magnitude homology
- Applications

Basics of persistent homology

Let X be a metric space. The Vietoris-Rips complex of X is* the functor

$$VR(X) : ([0,\infty], \leq) \to \mathbf{sSet}$$

given by the following data:

- For each $r \in [0,\infty]$, and for each $n \in \mathbb{N}$, $VR(X)(r)_n$ is the set of $(n+1)$-tuples $(x_0, \ldots, x_n)$ for which $d_X(x_i, x_j) \leq r$.
- $\partial_i : (x_0, \ldots, x_n) \mapsto (x_0, \ldots, \hat{x}_i, \ldots, x_n)$
  $\sigma_i : (x_0, \ldots, x_i, \ldots, x_n) \mapsto (x_0, \ldots, x_i, x_i, \ldots, x_n)$
- For $r \leq s$, $VR(X)(r) \to VR(X)(s)$ is the evident inclusion.

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We can take homology of $VR(X)$ pointwise along $[0, \infty]$; this is the persistent homology of X.

*This is kind of a lie, but is close enough to the truth for this talk; the full details are spelled out in the paper

Basics of magnitude homology

Let X be a metric space. The magnitude complex of X is the functor

$$M(X) : ([0,\infty], \leq) \to \mathbf{sSet}$$

given by the following data:

- For each $r \in [0,\infty]$, and for each $n \in \mathbb{N}$, $M(X)(r)_n$ is the set of $(n+1)$-tuples $(x_0, \ldots, x_n)$ for which $\sum_{1 \leq i \leq n} d_X(x_{i-1}, x_i) \leq r$.
- $\partial_i : (x_0, \ldots, x_n) \mapsto (x_0, \ldots, \hat{x}_i, \ldots, x_n)$
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Take free abelian groups to get $\overline{M}(X) : ([0,\infty]) \to \mathbf{sAb}$, and define $\operatorname{Loc}\overline{M}(X) : ([0,\infty], \leq) \to \mathbf{sSet}$ by

$$\operatorname{Loc}\overline{M}(X)(r) = \overline{M}(X)(r) / \bigcup_{s<r} \overline{M}(X)(s)$$

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Take pointwise homology of $\operatorname{Loc}\overline{M}(X)$ pointwise along $[0,\infty]$ to get the magnitude homology of X.

Frames and quantales

$\mathbb{R} = ([0, \infty], \geq)$ is an example of a *frame*:

- has meets and joins;
- joins commute with finite meets.

For each $1 \leq p \leq \infty$, there is a monoidal structure $+_p$ on $\mathbb{R}$:

$$a +_p b := (a^p + b^p)^{\frac{1}{p}}$$
$$a +_\infty b := \max(a, b)$$

which commutes with joins, and for which the unit is the terminal object $(0 \in \mathbb{R})$. Thus each $(\mathbb{R}, +_p)$ is an (*affine*) *quantale*.

- Every frame has a "default" quantale structure, where the monoidal structure is given by $\wedge$.
- For each $1 \leq p \leq q \leq \infty$, and for each $a, b \in \mathbb{R}$, we have $a +_p b \geq a +_q b$.

Notation

We will frequently switch between speaking of a quantale $\mathcal{V}$ and its opposite $\mathcal{V}^{\mathrm{op}}$ (which is frequently the "correct" ordering). We fix terminology:

- $\leq$ refers to the ordering on $\mathcal{V}^{\mathrm{op}}$
- inf (or min) refers to meets in $\mathcal{V}^{\mathrm{op}}$ (i.e. joins $\bigvee$ in $\mathcal{V}$)
- sup (or max) refers to joins in $\mathcal{V}^{\mathrm{op}}$ (i.e. meets $\bigwedge$ in $\mathcal{V}$)
- 0 refers to initial object of $\mathcal{V}^{\mathrm{op}}$, and ∞ to terminal object of $\mathcal{V}^{\mathrm{op}}$

Example/justification:

- $\mathbb{R} = ([0, \infty], \geq)$, so $\mathbb{R}^{\mathrm{op}} = ([0, \infty], \leq)$
- $a \wedge b$ is the meet of a, b in $\mathbb{R}$, thus the join of a, b in $\mathbb{R}^{\mathrm{op}}$, so $a \wedge b = \max(a, b) = a +_\infty b$.
- $a + b \geq \cdots \geq a +_2 b \geq \cdots \geq a +_\infty b = \max(a, b) = a \wedge b$

Graphs enriched over frames

Let $\mathcal{V}$ be a frame. Then a $\mathcal{V}$-graph X is given by the following data:

- A set of vertices, also denoted by X;
- For each pair $a, b \in X$, an object $X(a, b)$ of $\mathcal{V}$

and given X, Y two $\mathcal{V}$-graphs, a morphism $f : X \to Y$ is specified by

- A function (denoted $f : X \to Y$ by abuse) from the set of vertices of X to the set of vertices of Y, such that
- For all $a, b \in X$, $X(a, b) \geq Y(fa, fb)$.

Then $\mathcal{V}$-graphs and morphisms between them assemble into a category $\mathcal{V}$-**Gph**.

$\mathcal{V}$-**Gph** is complete and cocomplete.

Categories enriched over quantales

Let $(\mathcal{V}, \otimes, 0)$ be a quantale. Then a $(\mathcal{V}, \otimes)$-category X is given by the following data:

- A set of vertices, also denoted by X;
- For each pair $a, b \in X$, an object $X(a, b)$ of $\mathcal{V}$ such that
  - For all $a, b, c \in X$, we have $X(a, b) \otimes X(b, c) \geq X(a, c)$
  - For all $a \in X$, we have $X(a, a) = 0$.

A $(\mathcal{V}, \otimes)$-category is in particular a $\mathcal{V}$-graph, so declare $(\mathcal{V}, \otimes)$-**Cat** to be the full subcategory of $\mathcal{V}$-graphs on the $(\mathcal{V}, \otimes)$-categories.

$(\mathcal{V}, \otimes)$-**Cat** is complete and cocomplete. In fact, $(\mathcal{V}, \otimes)$-**Cat** is closed under taking limits in $\mathcal{V}$-**Gph**, so...

Categories enriched over quantales (cont'd)

The inclusion

$$\mathcal{I} : (\mathcal{V}, \otimes)\text{-}\mathbf{Cat} \hookrightarrow \mathcal{V}\text{-}\mathbf{Gph}$$

has a left adjoint $\mathcal{F} : \mathcal{V}\text{-}\mathbf{Gph} \to (\mathcal{V}, \otimes)\text{-}\mathbf{Cat}$.

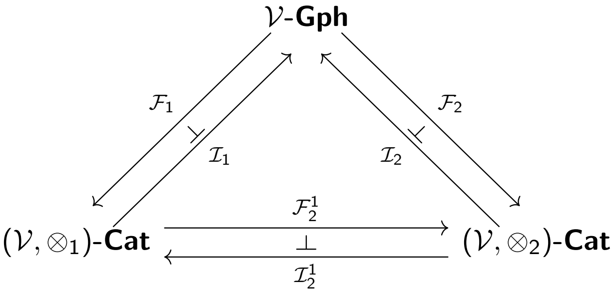
Given $X \in \mathcal{V}\text{-}\mathbf{Gph}$, $\mathcal{F}(X) \in (\mathcal{V}, \otimes)\text{-}\mathbf{Cat}$ has:

- The same vertices as X;
- $\mathcal{F}(X)(a, b) = \inf_{\substack{n \in \mathbb{N} \\ x_0, \ldots, x_n}} \left(\bigotimes_{1 \leq i \leq n} X(x_{i-1}, x_i) \right)$ with $x_0 = a$, $x_n = b$

Categories enriched over quantales (cont'd)

Let $\mathcal{V}$ be a frame, and let $(\mathcal{V}, \otimes_1)$ and $(\mathcal{V}, \otimes_2)$ be quantales with $\otimes_1 \geq \otimes_2$.

Then any $(\mathcal{V}, \otimes_2)$-category is also a $(\mathcal{V}, \otimes_1)$-category, so we have a full embedding $\mathcal{I}_2^1 : (\mathcal{V}, \otimes_2)$-**Cat** $\to (\mathcal{V}, \otimes_1)$-**Cat**. We have the following commuting diagram of adjunctions:



Simplices

Let $\mathcal{V}$ be a frame. For each $n \in \mathbb{N}$ and for each $(r_1, \ldots, r_n) \in \mathcal{V}^n$, we define $\Gamma^n(r_1, \ldots, r_n) \in \mathcal{V}$-**Gph** as:

- $\Gamma^n(r_1, \ldots, r_n)$ has $n+1$ vertices $x_0, \ldots, x_n$
- $$\Gamma^n(r_1, \ldots, r_n)(x_i, x_j) = \begin{cases} 0 & \text{if } i = j \\ r_j & \text{if } i = j-1 \\ \infty & \text{otherwise} \end{cases}$$

$\Gamma^n(r_1, \ldots, r_n)$ is "the n-spine with lengths $r_1, \ldots, r_n$"

Simplices (cont'd)

By $\Delta^n_\otimes(r_1, \ldots, r_n)$ we denote either $\mathcal{F}\Gamma^n(r_1, \ldots, r_n) \in (\mathcal{V}, \otimes)$-**Cat** or $\mathcal{I}\mathcal{F}\Gamma^n(r_1, \ldots, r_n) \in \mathcal{V}$-**Gph**. Explicitly:

$$\Delta^n_\otimes(r_1, \ldots, r_n)(x_i, x_j) = \begin{cases} \bigotimes_{i+1 \leq k \leq j} r_k & \text{if } i \leq j \\ \infty & \text{otherwise} \end{cases}$$

For $(r_1, \ldots, r_n) \leq (s_1, \ldots, s_n)$ there is a map

$$\sigma : \Delta^n_\otimes(s_1, \ldots, s_n) \to \Delta^n_\otimes(r_1, \ldots, r_n)$$

which is the identity on vertices.

Given $(\mathcal{V}, \otimes_1)$ and $(\mathcal{V}, \otimes_2)$ with $\otimes_1 \geq \otimes_2$, the (units of the) previous adjunctions gives the diagram

$$\begin{array}{ccc} & \Gamma^n(r_1, \ldots, r_n) & \\ \swarrow & & \searrow \\ \Delta^n_{\otimes_1}(r_1, \ldots, r_n) & \longrightarrow & \Delta^n_{\otimes_2}(r_1, \ldots, r_n) \end{array}$$

A nerve functor for each quantale

Let $(\mathcal{V}, \otimes)$ be a quantale. Let $X \in \mathcal{V}$-**Gph**. For each $n \in \mathbb{N}$, we have a functor $\widetilde{X}_n : (\mathcal{V}^{\mathrm{op}})^n \to$ **Set** where

- $\widetilde{X}_n(r_1, \ldots, r_n) := \mathcal{V}\text{-}\mathbf{Gph}\left(\Delta^n_\otimes(r_1, \ldots, r_n), X\right)$ and;
- $\widetilde{X}_n\left((r_1, \ldots, r_n) \leq (s_1, \ldots, s_n)\right)$ is the precomposition

$$\mathcal{V}\text{-}\mathbf{Gph}(\Delta^n_\otimes(r_1, \ldots, r_n), X) \xrightarrow{\sigma^*} \mathcal{V}\text{-}\mathbf{Gph}(\Delta^n_\otimes(s_1, \ldots, s_n), X)$$

by the evident map $\sigma : \Delta^n_\otimes(s_1, \ldots, s_n) \to \Delta^n_\otimes(r_1, \ldots, r_n)$.

$\widetilde{X}_n(r_1, \ldots, r_n)$ is the set of $(n+1)$-tuples $(x_0, \ldots, x_n)$ of vertices of X such that for each $0 \leq i \leq j \leq n$,

$$X(x_i, x_j) \leq \bigotimes_{i+1 \leq k \leq j} r_k$$

A nerve functor for each quantale (cont'd)

There is a functor $\bigotimes : (\mathcal{V}^{\mathrm{op}})^n \to \mathcal{V}^{\mathrm{op}}$ given by $(r_1, \ldots, r_n) \mapsto \bigotimes_i r_i$. Then we take $X_n := \mathrm{Lan}_{\otimes} \widetilde{X}_n$, so we have

- $X_n(r)$ is the set of $(n+1)$-tuples $(x_0, \ldots, x_n)$ of vertices of X such that $\exists (r_1, \ldots, r_n) \in \mathcal{V}^n$ for which $\bigotimes_{1 \le i \le n} r_i \le r$, and for each $0 \le i \le j \le n$,
$$X(x_i, x_j) \le \bigotimes_{i+1 \le k \le j} r_k$$
- $X_n(r \le s)$ is the evident inclusion.

A nerve functor for each quantale (cont'd)

Given $X \in \mathcal{V}$-**Gph**, the assignment

$$[n] \mapsto X_n$$

extends to a functor

$$N_{(\mathcal{V},\otimes)}(X) : \Delta^{\mathrm{op}} \to \mathrm{PSh}(\mathcal{V}).$$

where the i^{th} face (/degeneracy) map omits (/repeats) the i^{th} vertex.

$N_{(\mathcal{V},\otimes)}(X)$ is equivalently

- a functor $(\Delta^{\mathrm{op}} \times \mathcal{V}^{\mathrm{op}}) \to$ **Set**
- an object of $\mathrm{sPSh}(\mathcal{V})$

A nerve functor for each quantale (cont'd)

This construction is natural in X, so we have a functor

$$N_{(\mathcal{V},\otimes)} : \mathcal{V}\text{-}\mathbf{Gph} \to \mathsf{sPSh}(\mathcal{V})$$

- $N_{(\mathbb{R},\max)} : \mathbb{R}\text{-}\mathbf{Gph} \to \mathsf{sPSh}(\mathbb{R})$ is the Vietoris-Rips complex
- $N_{(\mathbb{R},+)} : \mathbb{R}\text{-}\mathbf{Gph} \to \mathsf{sPSh}(\mathbb{R})$ is (extends) the magnitude complex
- There is a natural transformation $N_{(\mathbb{R},\max)} \Longrightarrow N_{(\mathbb{R},+)}$
  - This is due to the canonical maps $\Delta^n_+(r_1,\ldots,r_n) \to \Delta^n_{\max}(r_1,\ldots,r_n)$

By $\overline{N}_{(\mathcal{V},\otimes)} : \mathcal{V}\text{-}\mathbf{Gph} \to \mathbf{sAb}^{\mathcal{V}^{\mathrm{op}}}$ denote $N_{(\mathcal{V},\otimes)}$ postcomposed with the free abelian group functor.

Localization along $\mathbb{R}^{\mathrm{op}}$

To each $r \in \mathbb{R}^{\mathrm{op}}$ assign a sieve J_r (i.e. downward closed subposet of $\mathbb{R}^{\mathrm{op}}$), in a way such that $r \leq s$ implies $J_r \subseteq J_s$.

For each $A \in \mathbf{sAb}^{\mathbb{R}^{\mathrm{op}}}$, define $\mathrm{Loc}_J A \in \mathbf{sAb}^{\mathbb{R}^{\mathrm{op}}}$ by

$$\mathrm{Loc}_J(A)(r) := A(r)/A(J_r)$$

where $A(J_r)$ is the union of the images $A(s) \to A(r)$ for all $s \in J_r$.

$$\mathrm{Loc}_J : \mathbf{sAb}^{\mathbb{R}^{\mathrm{op}}} \to \mathbf{sAb}_J^{\mathbb{R}^{\mathrm{op}}}$$

is a reflection onto the full subcategory $\mathbf{sAb}_J^{\mathbb{R}^{\mathrm{op}}}$ of the *J-local* objects of $\mathbf{sAb}^{\mathbb{R}^{\mathrm{op}}}$, i.e. those $A \in \mathbf{sAb}^{\mathbb{R}^{\mathrm{op}}}$ for which $\mathrm{Loc}_J(A) \cong A$.

So there is a natural transformation $1 \Longrightarrow \iota_J \circ \mathrm{Loc}_J$.

If for each $r \in \mathbb{R}^{\mathrm{op}}$ we take $J_r = [0, r)$ then magnitude homology is pointwise homology of $\mathrm{Loc}_J \overline{N}_{(\mathcal{V},\otimes)}$.

Comparison: persistent and magnitude homology

Persistent homology is

$$\mathbb{R}\text{-}\mathbf{Gph} \xrightarrow{\overline{N}_{(\mathbb{R},\max)}} \mathbf{sAb}^{\mathbb{R}^{\mathrm{op}}} \xrightarrow{H_\bullet} \mathbf{Ab}^{\mathbb{R}^{\mathrm{op}}}$$

Magnitude homology is

$$\mathbb{R}\text{-}\mathbf{Gph} \xrightarrow{\overline{N}_{(\mathbb{R},+)}} \mathbf{sAb}^{\mathbb{R}^{\mathrm{op}}} \xrightarrow{\mathrm{Loc}_J} \mathbf{sAb}_J^{\mathbb{R}^{\mathrm{op}}} \xrightarrow{\iota_J} \mathbf{sAb}^{\mathbb{R}^{\mathrm{op}}} \xrightarrow{H_\bullet} \mathbf{Ab}^{\mathbb{R}^{\mathrm{op}}}$$

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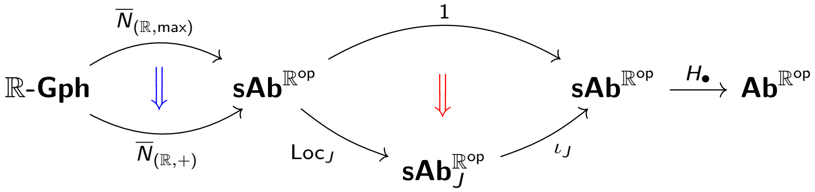

from persistent homology to magnitude homology.

Some applications

What follow are not really direct corollaries, but observations that the quantalic perspective naturally suggests.

- Persistent homology is an indicator of the failure of a metric space to be an ultrametric space.
- $H_1 \operatorname{Loc}_J \overline{N}_{(\mathbb{R},+_p)}$ measures (failure of) approximate collinearity.
- Magnitude homology applied to automata detects "cost-primitive" pairs of states.

A fact about magnitude homology

Let X be a metric space.

It is a result of Leinster and Shulman that for each $r > 0$, $H_1 \operatorname{Loc}_J \overline{N}_{(\mathbb{R},+)}(X)(r)$ (i.e. the 1^{st} magnitude homology group of X at scale r) is freely generated by ordered pairs (a, b) of points in X such that $d_X(a, b) = r$ and there exists no $c \in X$ such that $d_X(a, c) + d_X(c, b) = d_X(a, b)$.

That is, the 1^{st} magnitude homology group is freely generated by pairs of points between which there is no interpolating point.

Approximate collinearity

Let X be a metric space, and let a, b be distinct points of X.

Say that a point c *p-interpolates between a and b* when $a \neq c \neq b$ and there exist $r, s \in [0, \infty]$ such that

- $d_X(a, c) \leq r$
- $d_X(c, b) \leq s$
- $d_X(a, b) = (r^p + s^p)^{\frac{1}{p}}$

For $1 \leq p \leq q \leq \infty$, if a point p-interpolates between a and b then it q-interpolates as well.

Approximate collinearity (cont'd)

The same reasoning as in the proof of Leinster and Shulman shows that for $p \in [1, \infty)$, $H_1 \operatorname{Loc}_J \overline{N}_{(\mathbb{R},+_p)}(X)(r)$ is freely generated by ordered pairs (a, b) of points in X such that

- $d_X(a, b) = r$, and;
- There is no p-interpolating point between a and b.

Thus for any pair (a, b) of points in X, the quantity

$$p_{a,b} = \inf\{p \in [1, \infty) \mid (a, b) \text{ is trivial in } H_1 \operatorname{Loc}_J \overline{N}_{(\mathbb{R},+_p)}(X)(d_X(a, b))\}$$

indicates the existence of approximately collinear points between a and b (where nonexistence gives $p_{a,b} = \infty$); a lower value of $p_{a,b}$ indicates better approximation to collinearity.

Magnitude homology of automata

Given a monoid M, its powerset $\mathcal{P}(M)$ has the structure of a quantale, where for $A, B \in \mathcal{P}(M)$

$$A \otimes B = \{a \cdot b \mid a \in A \subseteq M, b \in B \subseteq M\}$$

where $\cdot$ is the monoid multiplication. The unit of $\mathcal{P}(M)$ is $\{e\}$ where e is the identity element of M.

An *automaton with inputs from M* is a $\mathcal{P}(M)$-category X. The vertices of X are the states, and the hom-object $X(a, b)$ is the set of elements of M which take a to b.

Magnitude homology of automata (cont'd)

We ask for a *cost function for M*, i.e. a lax monoidal functor $\mathbf{c} : M \to (\mathbb{R}, +)$ where M is considered as a discrete monoidal category:

- $\mathbf{c}(m_2) + \mathbf{c}(m_1) \geq \mathbf{c}(m_2 \cdot m_1)$
- $0 \geq \mathbf{c}(\{e\})$, i.e. $\mathbf{c}(\{e\}) = 0$

This induces a cost function $\mathbf{C} : \mathcal{P}(M) \to \mathbb{R}$ given by

$$\mathbf{C}(A) = \inf_{a \in A} \mathbf{c}(a)$$

which is strong monoidal if $\mathbf{c}$ is. Then for any automaton X we get an induced $(\mathbb{R}, +)$-category $\mathbf{C}(X)$ with the same vertices and $\mathbf{C}(X)(a, b) = \mathbf{C}\left(X(a, b)\right)$.

Magnitude homology of automata (cont'd)

A pair (a, b) of states (vertices) of the automaton X is called *cost-primitive* if there exists no state c such that $a \neq c \neq b$ and $\mathbf{C}(X)(a, b) = \mathbf{C}(a, c) + \mathbf{C}(X)(c, b)$.

If $\mathbf{C}(X)$ is "strict", i.e. there are no distinct pairs of points at distance 0 (meaning no states are connected in X by a 0-cost transition), then Leinster and Shulman's result applies:

- If (a, b) is a generator of 1^{st} magnitude homology then there is at least one transition taking a to b which does not occur as a composite of state transitions, and the cost of this transition is strictly less than the cost of any such composite transitions.
- If $\mathbf{c}$ has discrete range, then the converse is true: if there is at least one $m \in X(a, b)$ such that $\mathbf{c}(m)$ is less than the cost of all possible composite transitions from a to b, then (a, b) is cost-primitive and thus a generator of 1^{th} magnitude homology.

Thanks for listening!

References are the same as in:

- Simon Cho, *Quantales, persistence, and magnitude homology*, arXiv:1910.02905, 2019