An algebraic proof of the Frobenius condition for cubical sets

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QMS from a premodel

Definition

A *premodel* on a topos $\mathcal{E}$ consists of $(\Phi, \mathbb{I}, \mathrm{V})$ where:

- Φ is a representable class of monos $\Phi \hookrightarrow \Omega$ ...
- $\mathbb{I}$ is an interval $1 \rightrightarrows \mathbb{I}$ ...
- $\dot{\mathrm{V}} \to \mathrm{V}$ is a universe of *small families* ...

At CMU I sketched:

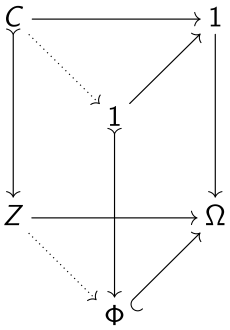
Construction

From a premodel $(\Phi, \mathbb{I}, \mathrm{V})$ *one can construct a QMS on* $\mathcal{E}$*.*

Today I will show that the resulting QMS is *right proper*.
This only uses Φ and $\mathbb{I}$.

1. The cofibration awfs $(\mathcal{C}, \mathsf{TFib})$

The monos $C \rightarrowtail Z$ classified by $\Phi \hookrightarrow \Omega$ are the *cofibrations* $\mathcal{C}$.



These are closed under pullbacks.

1. The cofibration awfs $(\mathcal{C}, \mathsf{TFib})$

The generic cofibration $1 \rightarrowtail \Phi$ determines a polynomial endofunctor,

$$X^+ := \sum_{\varphi : \Phi} X^\varphi .$$

This is a (fibered) monad,

$$+ : \mathcal{E} \longrightarrow \mathcal{E}$$

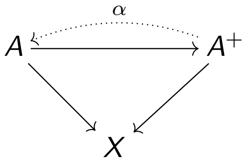
because of ...

1. The cofibration awfs $(\mathcal{C}, \mathsf{TFib})$

In each slice $\mathcal{E}/X$, the algebras (A, α) for the underlying pointed endofunctor,

$$+_X : \mathcal{E}/X \longrightarrow \mathcal{E}/X$$

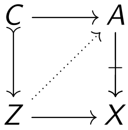
are the *trivial fibrations*.



They form the right class of the *cofibration awfs* $(\mathcal{C}, \mathsf{TFib})$.

1. The cofibration awfs ($\mathcal{C}$, TFib)

The algebra structures on a trivial fibration correspond uniquely to *uniform right lifting structures* against the cofibrations:



That is, a choice of fillers, ...

1. The cofibration awfs ($\mathcal{C}$, TFib)

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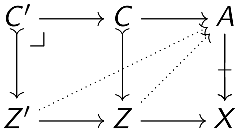


That is, a choice of flllers, that are *coherent*.

1. The cofibration awfs ($\mathcal{C}$, TFib)

The trivial fibrations are closed under all pullbacks, because they are "on the right". But because their left class $\mathcal{C}$ is also closed under all pullbacks, we have:

Proposition

The trivial fibrations are closed under all pushforwards.

This follows by a standard "adjoint lemma":

Lemma

Given a wfs $(\mathcal{L}, \mathcal{R})$ and a base change $f^* \dashv f_* : \mathcal{E}/X \longrightarrow \mathcal{E}/Y$,

$$f^* \text{ preserves } \mathcal{L} \quad \text{iff} \quad f_* \text{ preserves } \mathcal{R}.$$

2. The fibration awfs $(\mathsf{TCof}, \mathcal{F})$

For any map $u : A \to B$ in $\mathcal{E}$, the *Leibniz adjunction*

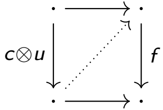
$$(-)\otimes u \;\dashv\; u \Rightarrow (-)$$

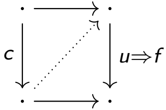
relates the pushout-product with u and the pullback-hom with u.

The functors $(- \otimes u) \dashv (u \Rightarrow -) : \mathcal{E}^{\mathbb{2}} \longrightarrow \mathcal{E}^{\mathbb{2}}$ also satisfy

$$(c \otimes u) \boxslash f \;\Leftrightarrow\; c \boxslash (u \Rightarrow f)$$

with respect to the diagonal filling relation $c \boxslash f$.





This holds also for uniform filling *structures*.

2. The fibration awfs $(\mathsf{TCof}, \mathcal{F})$

We then define the fibrations in terms of the trivial fibrations,

$$f \in \mathcal{F} \quad \text{iff} \quad \delta \Rightarrow f \in \mathsf{TFib}$$

using the pullback-hom $\delta \Rightarrow f$ with the *generic point* $\delta : 1 \to \mathbb{I}$ in the slice category $\mathcal{E}/\mathbb{I}$.

Definition

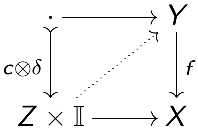
A map $f : Y \to X$ is a *fibration* if $\delta \Rightarrow f$ is a trivial fibration in $\mathcal{E}/\mathbb{I}$. Equivalently, by adjointness, $f \in \mathcal{F}$ iff $c \otimes \delta \boxslash f$,

$$\begin{array}{ccc} Z +_C (C \times \mathbb{I}) & \longrightarrow & Y \\ {\scriptstyle c \otimes \delta} \big\downarrow & \nearrow & \big\downarrow {\scriptstyle f} \\ Z \times \mathbb{I} & \longrightarrow & X \end{array}$$

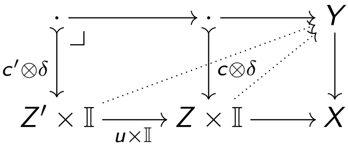
for all cofibrations $c : C \rightarrowtail Z$.

2. The fibration awfs $(\mathsf{TCof}, \mathcal{F})$

A *fibration structure* on $f : Y \to X$ thus corresponds to a *right lifting structure* against all $c \otimes \delta$,



that is *uniform* with respect to all base changes $u : Z' \longrightarrow Z$,



2. The fibration awfs $(\mathsf{TCof}, \mathcal{F})$

Proposition

There is an awfs $(\mathsf{TCof}, \mathcal{F})$ *with these "uniform fibrations" as* $\mathcal{F}$.

Like the trivial fibrations, the fibrations are closed under pullbacks because they are "on the right". But unlike the trivial fibrations, they are *not* closed under all pushforwards. However, we do have:

Proposition

The fibrations are closed under pushforward along fibrations.

This is the main result to be shown below.

3. Frobenius

Proposition

The fibrations are closed under pushforward along fibrations.

Note that the "adjoint lemma" then implies:

Corollary (Frobenius)

The trivial cofibrations are closed under pullback along fibrations.

But since $\mathcal{W} = \mathsf{TFib} \circ \mathsf{TCof}$, and TFib is closed under pullbacks:

Proposition

$\mathcal{W}$ is closed under pullback along $\mathcal{F}$, i.e. the QMS is **right proper***.*


RIGHT PROPER
BREWING COMPANY

3. Frobenius

Again, Frobenius says that trivial cofibrations pull back along fibrations,

$$\begin{array}{ccc} C' & \overset{\sim}{\rightarrowtail} & A \\ \downarrow & \lrcorner & \downarrow\!\!\downarrow \\ C & \overset{\sim}{\rightarrowtail} & X\,. \end{array}$$

This is used for the interpretation of Id-types.

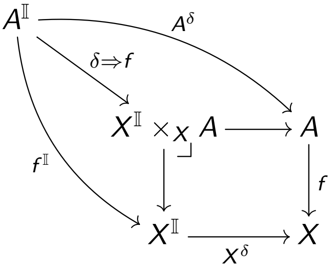
By the adjoint lemma, it is equivalent that fibrations push forward along fibrations,

$$\begin{array}{ccc} B & \twoheadrightarrow & A \\ & & \downarrow\!\!\downarrow \\ \Pi_A B & \twoheadrightarrow & X\,. \end{array}$$

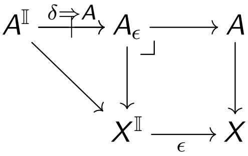
This is used for the interpretation of Π-types.

3. Frobenius

Recall that $f : A \to X$ is a fibration iff $\delta \Rightarrow f$ is a trivial fibration:



We indicate this briefly as follows:



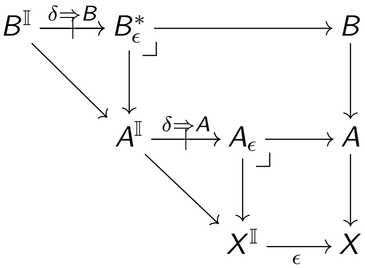
3. Frobenius

Proposition

Fibrations push forward along fibrations.

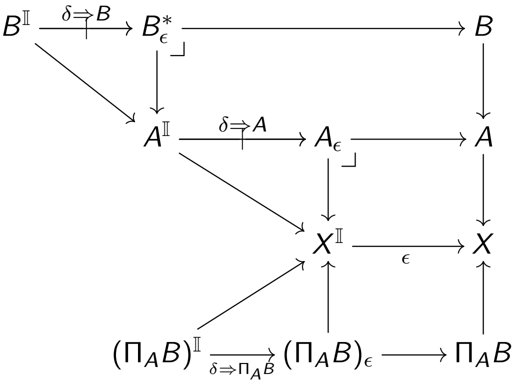
Proof.

Consider fibrations $B \twoheadrightarrow A \twoheadrightarrow X$.
Thus we have:



3. Frobenius

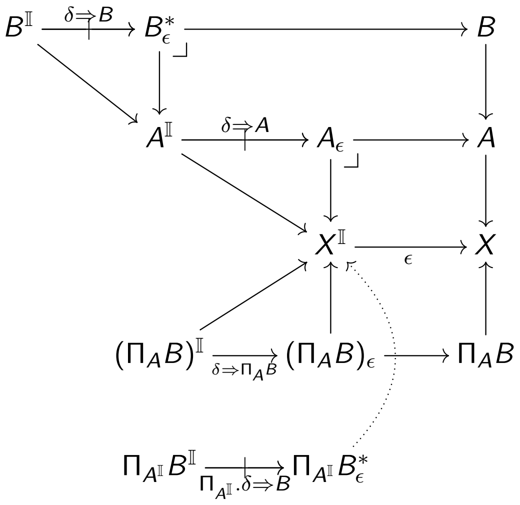
Taking the pushforward of the right column yields $\Pi_A B \longrightarrow X$.



We want to show that $\delta \Rightarrow \Pi_A B$ is a trivial fibration.

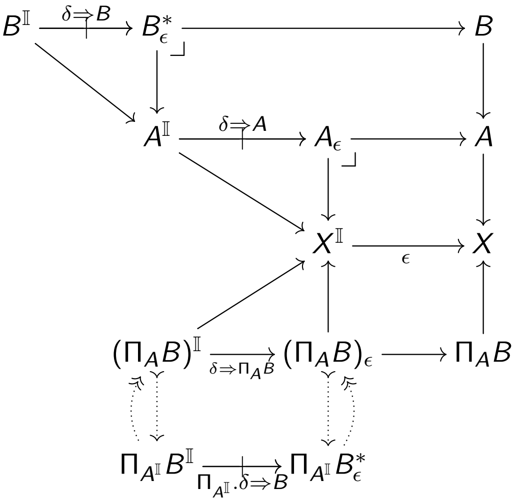
3. Frobenius

The pushforward of $\delta \Rightarrow B$ along $A^{\mathbb{I}} \longrightarrow X^{\mathbb{I}}$ is a trivial fibration over $X^{\mathbb{I}}$, since these are closed under all pushforwards.



3. Frobenius

One then shows that there is a retraction of $\Pi_{A^{\mathbb{I}}}.\delta \Rightarrow B$ onto $\delta \Rightarrow \Pi_A B$ over $X^{\mathbb{I}}$.

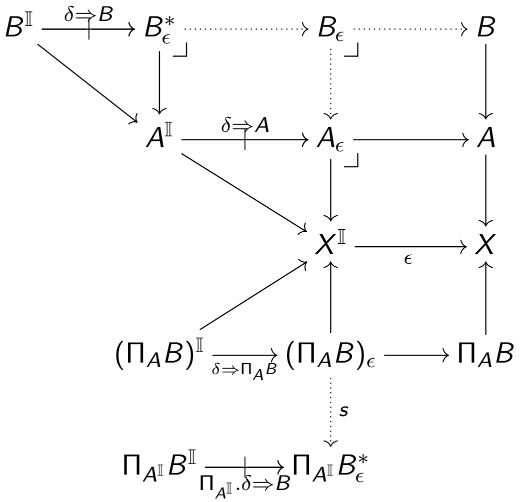


□

3. Frobenius

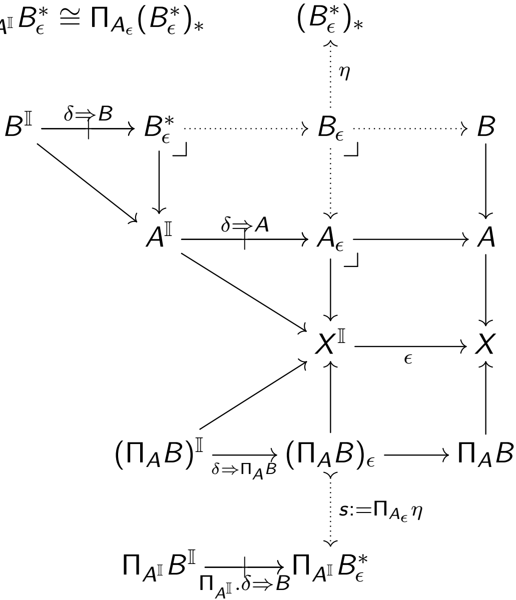
For example, to get $s : (\Pi_A B)_\epsilon \longrightarrow \Pi_{A^{\mathbb{I}}} B^*_\epsilon$ first interpolate B_ϵ so

$$(\Pi_A B)_\epsilon \cong \Pi_{A_\epsilon} B_\epsilon$$



3. Frobenius

Then use $\Pi_{A^{\mathbb{I}}} B_\epsilon^* \cong \Pi_{A_\epsilon}(B_\epsilon^*)_*$



□

4. References

This “algebraic” proof of Frobenius is derived from a type theoretic one due to Thierry Coquand:

- Cohen, Coquand, Huber, Mörtberg:
  Cubical Type Theory: A constructive interpretation of the univalence axiom, TYPES 2015.

Also see:

- Awodey: A Quillen model structure on cartesian cubical sets, github.com/awodey/math/qms (2019)