

# Pseudofunctors and simplicial categories

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Overview of the  
problem

Enrichment  
through variation

Our higher  
dimensional  
version

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  - ▶ Variable *base* of enrichment
  - ▶ Variable *strength* of enrichment
  - ▶ But in a usable format

# Outline

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We can study

- ▶  $s\mathcal{E}$ : simplicial objects in  $\mathcal{E}$ ,
- ▶  $\mathbf{qCat}(\mathcal{E})$ : internal quascategories in  $\mathcal{E}$ , and
- ▶  $\mathbf{Segal}(\mathcal{E})$ : internal Segal categories in  $\mathcal{E}$ .

# Application

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Higher dimensional versions:

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**Conclusion:** we are looking for functors from a simplicially enriched category to a 2-category.

# Graphs first

Recall the adjunction  $\tau_1 \dashv N$

- ▶  $N : \mathbf{sSet} \rightarrow \mathbf{Cat}$ ,
- ▶  $\tau_1 : \mathbf{Cat} \rightarrow \mathbf{sSet}$ .



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### Lemma

For any adjunction  $F \dashv U$  with  $F : \mathcal{A} \rightleftarrows \mathcal{B}$  and any category  $\mathbb{D}$ , there is an induced adjunction

$$F_* \dashv U_* \quad \text{with} \quad F_* : [\mathbb{D}, \mathcal{A}] \rightleftarrows [\mathbb{D}, \mathcal{B}] : U_*.$$

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Apply with  $\mathbb{D} = m \rightrightarrows o$  to the adjunction above to get  $(\tau_1)_* \dashv N_*$  with

$$(\tau_1)_* : \mathbf{Graph}(\mathbf{sSet}) \rightleftarrows \mathbf{Graph}(\mathbf{Cat}) : N_*.$$

# From graphs to enriched categories

Our lemma constructing  $F_* \dashv U_*$  for graphs can be extended.

## Proposition

1. If  $P : \mathcal{A} \rightarrow \mathcal{B}$  is a lax monoidal functor between monoidal categories, then it induces a functor

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2. Under the same hypotheses,  $P$  induces a functor

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3. If  $F \dashv U$  is a monoidal adjunction, it induces an adjunction

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## Lemma

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**Refined question:** Can we extend our original functor  $\mathbf{qCat}(\mathcal{E}) \rightarrow \mathbf{Cat}$  to one of the equivalent kinds below?

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# In broad strokes

Gordon and Power compare two concepts:

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Gordon and Power compare two concepts:

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Think:

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The left are the enriched categories, while the right are the representations.

We fix a monoidal category  $\mathcal{V}$  over which to enrich.

For any enriched category  $\mathcal{C}$ , write  $\mathcal{C}_0$  for the underlying category.

# Tensors

## Definition

A  $\mathcal{V}$ -category  $\mathcal{C}$  is **tensoried** if

$$\mathcal{V}(v, \mathcal{C}(c, -)) : \mathcal{C} \rightarrow \mathcal{V}$$

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## Our examples

- ▶ Any closed monoidal category ( $\mathbf{Cat}$ ) is tensoried over itself
- ▶  $(\tau_1)_* \mathbf{qCat}(\mathcal{E})$  is also tensoried over  $\mathbf{Cat}$  using nerves



# Representations

## Definition

A  $\mathcal{V}$ -representation  $L$  is a category (unenriched!) with

- ▶ an action  $\star : \mathcal{V}_0 \times L \rightarrow L$  and
- ▶ natural iso's  $I \star c \cong c$ ,  $w \star (v \star c) \cong (w \otimes v) \star c$

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## Immediate consequence

$\mathcal{V}\text{-Rep} \cong \mathbf{Lax}(\Omega^{-1}\mathcal{V}, \mathbf{Cat})$  as 2-categories.

- ▶  $\mathbf{Lax}(\Omega^{-1}\mathcal{V}, \mathbf{Cat})$  is pseudofunctors, lax transformations, and modifications
- ▶  $\Omega^{-1}\mathcal{V}$  is the 1-object bicategory with  $\mathcal{V}$  as the single endo-hom-category

# The standard embedding

## Theorem (Gordon-Power)

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3. If  $\mathcal{V}$  is right closed, this 2-functor is full and faithful.
4. In this case, the essential image consists of all **closed representations**.

# Non-standard embeddings for dense subcategories

Gordon-Power explain a further generalization of this theory.

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and conditions under which this is full and faithful

- ▶ Can use  $\omega$  to detect whether  $\mathcal{C}$  has all tensors

## Back to our application

**Refined question:** Can we extend our original functor  $\mathbf{qCat}(\mathcal{E}) \rightarrow \mathbf{Cat}$  to one of the equivalent kinds below?

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Unfortunately, we don't seem to have such a thing: our maps of representations are constructed from universal properties, so the axioms only hold up to isomorphism.

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# A poorly stated theorem

## Theorem [G-Schäppi]

A weak map of  $\mathbf{Cat}$ -representations induces a weakly tensor-preserving pseudofunctor between 2-categories.

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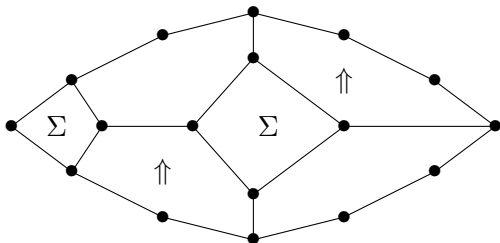
### Consequence

We get a pseudofunctor

$$(\tau_1)_* \mathbf{qCat}(\mathcal{E}) \rightarrow \mathbf{Cat}$$

using the dense subcategory version of the embedding.

# Thanks!



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