Pseudofunctors and simplicial categories

Nick Gurski
Case Western Reserve University

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   - Variable *strength* of enrichment
   - But in a usable format

Outline

Overview of the problem

Enrichment through variation

Our higher dimensional version

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We can study

- $s\mathcal{E}$: simplicial objects in $\mathcal{E}$,
- $\mathbf{qCat}(\mathcal{E})$: internal quasicategories in $\mathcal{E}$, and
- $\mathbf{Segal}(\mathcal{E})$: internal Segal categories in $\mathcal{E}$.

Application

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$$\mathbf{qCat}(\mathcal{E}) \to \mathbf{Cat}$$

by

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Conclusion: we are looking for functors from a simplicially enriched category to a 2-category.

Graphs first

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For any adjunction $F \dashv U$ with $F : \mathcal{A} \rightleftarrows \mathcal{B}$ and any category $\mathbb{D}$, there is an induced adjunction

$$F_* \dashv U_* \quad \text{with} \quad F_* : [\mathbb{D}, \mathcal{A}] \rightleftarrows [\mathbb{D}, \mathcal{B}] : U_*.$$

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Apply with $\mathbb{D} = m \rightrightarrows o$ to the adjunction above to get $(\tau_1)_* \dashv N_*$ with

$$(\tau_1)_* : \mathbf{Graph}(\mathbf{sSet}) \rightleftarrows \mathbf{Graph}(\mathbf{Cat}) : N_*.$$

From graphs to enriched categories

Our lemma constructing $F_* \dashv U_*$ for graphs can be extended.

Proposition

1. If $P : \mathcal{A} \to \mathcal{B}$ is a lax monoidal functor between monoidal categories, then it induces a functor

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3. If $F \dashv U$ is a monoidal adjunction, it induces an adjunction
$$F_* : \mathcal{A}\text{-}\mathbf{Cat} \rightleftarrows \mathcal{B}\text{-}\mathbf{Cat} : U_*.$$

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Refined question: Can we extend our original functor $\mathbf{qCat}(\mathcal{E}) \to \mathbf{Cat}$ to one of the equivalent kinds below?

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$$\updownarrow$$
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The left are the enriched categories, while the right are the representations.

We fix a monoidal category $\mathcal{V}$ over which to enrich.

For any enriched category $\mathcal{C}$, write $\mathcal{C}_0$ for the underlying category.

Tensors

Definition

A $\mathcal{V}$-category $\mathcal{C}$ is tensored if

$$\mathcal{V}\big(v, \mathcal{C}(c, -)\big) : \mathcal{C} \to \mathcal{V}$$

is representable. Write the representing object as $v \odot c \in \mathcal{C}$.

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Our examples

- Any closed monoidal category $(\mathbf{Cat})$ is tensored over itself
- $(\tau_1)_* \mathbf{qCat}(\mathcal{E})$ is also tensored over $\mathbf{Cat}$ using nerves

Representations

Definition

A $\mathcal{V}$-representation L is a category (unenriched!) with

- an action $\star : \mathcal{V}_0 \times L \to L$ and
- natural iso's $I \star c \cong c$, $w \star (v \star c) \cong (w \otimes v) \star c$

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Immediate consequence

$\mathcal{V}\text{-}\mathbf{Rep} \cong \mathbf{Lax}(\Omega^{-1}\mathcal{V}, \mathbf{Cat})$ as 2-categories.

- $\mathbf{Lax}(\Omega^{-1}\mathcal{V}, \mathbf{Cat})$ is pseudofunctors, lax transformations, and modifications
- $\Omega^{-1}\mathcal{V}$ is the 1-object bicategory with $\mathcal{V}$ as the single endo-hom-category

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3. If $\mathcal{V}$ is right closed, this 2-functor is full and faithful.
4. In this case, the essential image consists of all closed representations.

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  and conditions under which this is full and faithful
- Can use ω to detect whether $\mathcal{C}$ has all tensors

Back to our application

Refined question: Can we extend our original functor $\mathbf{qCat}(\mathcal{E}) \to \mathbf{Cat}$ to one of the equivalent kinds below?

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Unfortunately, we don't seem to have such a thing: our maps of representations are constructed from universal properties, so the axioms only hold up to isomorphism.

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A poorly stated theorem

Theorem [G-Schäppi]

A weak map of $\mathbf{Cat}$-representations induces a weakly tensor-preserving pseudofunctor between 2-categories.

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Consequence

We get a pseudofunctor

$$(\tau_1)_* \mathbf{qCat}(\mathcal{E}) \to \mathbf{Cat}$$

using the dense subcategory version of the embedding.

Thanks!