A Categorification of Group Cohomology

Michael Horst

The Ohio State University
horst.59@osu.edu
https://www.asc.ohio-state.edu/horst.59/

27 October 2019

Picard Categories

- Groupoid
- Symmetric monoidal
- Group-like:
  - For all X, there is a Y such that $X \otimes Y \cong I \cong Y \otimes X$
- Essential data:
  - $\pi_0(\mathscr{C}) = Obj(\mathscr{C}) \, / \cong$
  - $\pi_1(\mathscr{C}) = \mathscr{C}(I, I)$
  - $K : \pi_0(\mathscr{C}) \to \pi_1(\mathscr{C})$,
    $X \mapsto \beta_{X,X} \in \mathscr{C}(X \otimes X, X \otimes X) \cong \pi_1(\mathscr{C})$

Examples

- $Pic(R) := R\text{-}\mathbf{Mod}_{inv}^{\cong}$, for $R \in \mathbf{CRing}$
  - Note: $\pi_0(Pic(R)) = pic(R)$
- $\Pi_1 X$ for $X \in \Omega^3\mathbf{Top}$
- $\mathcal{Z}$, "Super Integers"
  - $Obj(\mathcal{Z}) = \mathbb{Z}, \;\; \mathcal{Z}(n,m) \cong \begin{cases} \mathbb{Z}/2, & \text{if } n = m \\ 0, & \text{else} \end{cases}$
  - Call $\mathcal{Z}(n,n) = \{\pm 1_n\}$
  - $(\beta : n+m \to m+n) = (-1_{n+m})^{nm}$

Free Picard Categories

Theorem (H)

The forgetful functor $U : \mathbf{Pic} \to \mathbf{Grpd}$ *has a left biadjoint given by*

$$\mathbb{Z}[\_] : \mathbf{Grpd} \to \mathbf{Pic}.$$

Specifically: For $\mathscr{G} \in \mathbf{Grpd}$ and $\mathscr{A} \in \mathbf{Pic}$,

$$\mathbf{Pic}(\mathbb{Z}[\mathscr{G}], \mathscr{A}) \simeq \mathbf{Grpd}(\mathscr{G}, \mathscr{A})$$

as Picard categories, pseudonatural in $\mathscr{G}$ and $\mathscr{A}$.

Tensoring over **Grpd**

Corollary (H)

Pic *is tensored over* **Grpd**

Specifically: For $\mathscr{G} \in \mathbf{Grpd}$ and $\mathscr{A} \in \mathbf{Pic}$, there exists $\mathscr{A}[\mathscr{G}] \in \mathbf{Pic}$ so that for all $\mathscr{B} \in \mathbf{Pic}$,

$$\mathbf{Pic}(\mathscr{A}[\mathscr{G}], \mathscr{B}) \simeq \mathbf{Grpd}(\mathscr{G}, \mathbf{Pic}(\mathscr{A}, \mathscr{B}))$$

pseudonaturally.

Proposition (H)

Pic *is cotensored over* **Grpd**

Picard Cohomology

Definition: Category of Modules over $\mathscr{G} \in \mathbf{Pic}$

For $\mathscr{G} \in \mathbf{Pic}$, $\mathscr{G}\text{-}\mathbf{Mod} := \mathrm{PsFunk}\left(\Sigma\mathscr{G}, \mathbf{Pic}\right)$

Definition: Picard Cohomology

For $\mathscr{G} \in \mathbf{Pic}$ and $\mathscr{M} \in \mathscr{G}\text{-}\mathbf{Mod}$, define

$$\mathscr{H}^n\left(\mathscr{G}; \mathscr{M}\right) := \mathbb{R}^n\Big(\mathscr{G}\text{-}\mathbf{Mod}\left(\mathscr{Z}_{\text{triv}}, \_\right)\Big)\left(\mathscr{M}\right)$$

i.e.

$$\mathscr{H}^n\left(\mathscr{G}; \mathscr{M}\right) = \mathrm{Ext}^n_{\mathscr{G}\text{-}\mathbf{Mod}}\left(\mathscr{Z}_{\text{triv}}, \mathscr{M}\right)$$

i.e.

$$\mathscr{H}^n\left(\mathscr{G}; \mathscr{M}\right) = \mathbb{L}^n\Big(\mathscr{G}\text{-}\mathbf{Mod}\left(\_, \mathscr{M}\right)\Big)\left(\mathscr{Z}_{\text{triv}}\right)$$

A first computation: $\mathscr{H}^m(\mathbb{Z}/n;\mathcal{Z})$

Proposition (H)

The chain complex

$$\dots \xrightarrow{N} \mathcal{Z}[\mathbb{Z}/n] \xrightarrow{x-1} \mathcal{Z}[\mathbb{Z}/n] \xrightarrow{N} \mathcal{Z}[\mathbb{Z}/n] \xrightarrow{x-1} \mathcal{Z}[\mathbb{Z}/n] \xrightarrow{0} 0 \xrightarrow{0} \dots$$

provides a projective resolution of

$$\dots \xrightarrow{0} 0 \xrightarrow{0} \mathcal{Z}_{triv} \xrightarrow{0} 0 \xrightarrow{0} \dots$$

Lemma (H)

Applying $\mathbb{Z}/n$-$\mathbf{Mod}(\_\,,\mathcal{Z}_{triv})$ *to the above yields*

$$0 \xrightarrow{0} 0 \xrightarrow{0} \mathcal{Z} \xrightarrow{0} \mathcal{Z} \xrightarrow{n} \mathcal{Z} \xrightarrow{0} \mathcal{Z} \xrightarrow{n} \dots$$

A first computation: $\mathscr{H}^m(\mathbb{Z}/n;\mathcal{Z})$

Theorem (H)

For n even,

$$\mathscr{H}^m(\mathbb{Z}/n;\mathcal{Z}) = \begin{cases} \mathcal{Z} \oplus \mathbb{Z}/2 & m = 0 \\ \mathbb{Z}/2 \oplus \Sigma(\mathbb{Z} \times \mathbb{Z}/2) & m = 1 \\ \mathcal{Z}/n \oplus \mathbb{Z}/2 & m > 1 \textit{ is even} \\ \mathbb{Z}/2 \oplus \Sigma(\mathbb{Z}/n \times \mathbb{Z}/2) & m > 1 \textit{ is odd} \end{cases}$$

For n odd,

$$\mathscr{H}^m(\mathbb{Z}/n;\mathcal{Z}) = \begin{cases} \mathcal{Z} & m = 0 \\ \Sigma(\mathbb{Z}) & m = 1 \\ \mathbb{Z}/n & m > 1 \textit{ is even} \\ \Sigma(\mathbb{Z}/n) & m > 1 \textit{ is odd} \end{cases}$$

A second computation: $\mathscr{H}^m(\mathbb{Z}/n;\mathbb{Z})$

Lemma (H)

Applying $\mathbb{Z}/n$-$\mathbf{Mod}(\_,\mathbb{Z}_{triv})$ *to the previous resolution yields*

$$0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \dots$$

Theorem (H)

For all n,

$$\mathscr{H}^m(\mathbb{Z}/n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & m = 0 \\ \Sigma(\mathbb{Z}) & m = 1 \\ \mathbb{Z}/n & m > 1 \text{ is even} \\ \Sigma(\mathbb{Z}/n) & m > 1 \text{ is odd} \end{cases}$$

Chain complexes of Picard categories

- Chain complex of Picard categories

$$\begin{tikzcd}
\cdots \arrow[r] & \mathscr{A}_{n-2} \arrow[r, "d_{n-2}"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=22, "\Uparrow \partial_{n-2}"] & \mathscr{A}_{n-1} \arrow[r, "d_{n-1}"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=22, "\Downarrow \partial_{n-1}"] & \mathscr{A}_{n} \arrow[r, "d_{n}"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=22, "\Uparrow \partial_{n}"] & \mathscr{A}_{n+1} \arrow[r, "d_{n+1}"] & \mathscr{A}_{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

- Compute cohomology using *relative* kernel and cokernel

$$\begin{tikzcd}
\mathrm{Ker}\,(F, \varphi) \arrow[r, "k"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=22, "\Downarrow \kappa"] & \mathscr{A} \arrow[r, "F"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=22, "\Uparrow \varphi"] & \mathscr{B} \arrow[r, "G"] & \mathscr{C}
\end{tikzcd}$$

- If $\mathscr{H}^n(\mathscr{A}_\bullet) \simeq 0$, call $\mathscr{A}_\bullet$ *relative exact* at $\mathscr{A}_n$

$\mathscr{G}$-modules vs. Picard Categories

Theorem (H)

Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{D}$ be bicategories, and assume that $\mathscr{A}$ has all bicategorical limits (dually, colimits) of shape $\mathscr{D}$. Then for any $F : \mathscr{D} \to \mathit{PsFunk}(\mathscr{B}, \mathscr{A})$, $\mathrm{Lim}\, F$ exists and all its data are computed objectwise. Dually, the same holds for $\mathrm{Colim}\, F$.

Proposition

Any biadjunction $F \dashv G$ between bicategories $\mathscr{A}$ and $\mathscr{B}$ yields for any bicategory $\mathscr{D}$ a biadjunction $F^{\mathscr{D}} \dashv G^{\mathscr{D}}$ between $\mathit{PsFunk}(\mathscr{D}, \mathscr{A})$ and $\mathit{PsFunk}(\mathscr{D}, \mathscr{B})$.

Projective Picard categories

Definition

$\mathscr{P} \in$ **Pic** is *projective* if for all

$$\begin{tikzcd} & \mathscr{P} \arrow[dl, dashed] \arrow[d, "H"] \arrow[dl, phantom, shift left=3, "\cong"] \\ \mathscr{B} \arrow[r, "G"'] & \mathscr{C} \end{tikzcd}$$

with G essentially surjective, such a lift exists.

- Problems:
  - No homological rephrasing
  - $\mathscr{P}$ is projective if and only if $\mathscr{P} = \mathbb{Z}^{\oplus\kappa} = \mathbb{Z}[\mathscr{G}]$ for *discrete* $\mathscr{G}$

Relative projective Picard categories

Definition (H)

$\mathscr{P} \in$ **Pic** is *relative projective* if for all

$$\begin{tikzcd}
 & & \mathscr{P} \arrow[dl, dashed] \arrow[d, "H"] \arrow[d, phantom, shift right=4ex, "\cong"] \\
\mathscr{A} \arrow[r, "F"'] \arrow[rr, bend right=40, "0"'] & \mathscr{B} \arrow[r, "G"'] \arrow[d, phantom, "\Downarrow\varphi"] & \mathscr{C} \\
 & {} &
\end{tikzcd}$$

with G essentially surjective and φ-full, such a lift exists.

- "φ-full" is a generalization of full

Relative projective Picard categories

Definition, rephrased

$\mathscr{P} \in$ **Pic** is *relative projective* if for all

$$\begin{tikzcd}
 & & \mathscr{P} \arrow[dl, dashed, "\cong"] \arrow[d, "H"] & & \\
\mathscr{A} \arrow[r, "F"'] \arrow[rr, bend right=40, "0"'] \arrow[rr, phantom, bend right=20, "\Downarrow\varphi"] & \mathscr{B} \arrow[r, "G"'] & \mathscr{C} \arrow[r, "0"'] & 0 \arrow[r, "0"'] & 0
\end{tikzcd}$$

with row relative exact at $\mathscr{C}$, such a lift exists.

Relative projective Picard categories

Theorem (H)

$\mathscr{P} \in \mathbf{Pic}$ is relative projective if and only if $\mathbf{Pic}(\mathscr{P}, \_)$ is relative exact.

- But not all free Picard categories are relative projective!

Proposition (H)

$\mathbb{Z}[\mathscr{G}]$ is relative projective if and only if for all $G \in \mathscr{G}$, $End(G)$ is free.

Proposition (H)

For free $A \in \mathbf{Ab}$, $\Sigma A \in \mathbf{Pic}$ is relative projective but not free.

Thank you$\sim$!

References

- J. C. Baez and A. D. Lauda, *Higher-dimensional algebra. v: 2-groups.*, Theory and Applications of Categories [electronic only] **12** (2004), 423–491.
- A. del Río, J. Martínez-Moreno, and E. Vitale, *Chain complexes of symmetric categorical groups.*, J. Pure Appl. Algebra **196** (2005), no. 2-3, 279–312. doi:10.1016/j.jpaa.2004.08.029
- M. Dupont, *Abelian Categories in dimension 2.*, Ph.D. thesis, Université catholique de Louvain, 2008. arXiv:0809.1760
- N. Gurski, N. Johnson, and A. M. Osorno, *Star product on Picard Categories.*, 2018, personal correspondence.
- N. Johnson and A. M. Osorno, *Modeling stable one-types.*, Theory Appl. Categ. **26** (2012), 520–537.
- M. Kapranov, *Supergeometry in mathematics and physics.*, 2015. arXiv:1512.07042
- T. Pirashvili, *On Abelian 2-categories and derived 2-functors.*, 2010. arXiv:1007.4138
- ______, *Projective and injective symmetric categorical groups and duality.*, Proc. Am. Math. Soc. **143** (2015), no. 3, 1315–1323. doi:10.1090/S0002-9939-2014-12354-9