The Explanatory Value of Category Theory

Ellen Lehet

University of Notre Dame

My Project

Motivation Category theory has been incredibly useful in contemporary mathematics. Its contribution to the progress and development of mathematics raises the question: *what about category theory allows it to be so useful?*

Aims
- Demonstrate that category theory is in many ways explanatory. It has a tendency to produce explanations.
- Use the case of category theory to inform the epistemology of mathematics.

Argument

P1 Category theory gives rise to a unique kind of structure, which I call *sophisticated structure*.

P2 The structure of category theory produces mathematical analogies and generalizations.

P3 The resulting analogies and generalizations produced are explanatory.

Conclusion Therefore, category theory has explanatory value.

Sophisticated Structure

Category theory's emphasis on arrows provides a unique kind *sophisticated structure*. Compare:

- Naive Structure
  - synonymous with Bourbakian structure (think structured set)
  - fundamentally set theoretic
- Sophisticated Structure
  - Involves relations of arrows, and
  - is not restricted to set theoretic constructions.

This unique kind of structure gives rises to generalizations and analogies.

Example

We can nicely see the difference between naive and sophisticated structure by comparing different ways of thinking of groups:

- $(G, +)$: presents a group as a set with additional structure
- BG: presents a group as a category, where arrows are not set-theoretic constructions

Generalization

Category theory introduces two kinds of generalizations:

1. General constructions
2. General results

General Constructions

Example 1: The Fundamental Groupoid

Definition

For a topological space X the fundamental groupoid is the category whose objects are the points of X, with arrows given by paths in X, and composition give by concatentation of paths.

- Allows us to abstract away from choice of basepoint.
- Provides new ways of proving important results in algebraic topology, such as the van Kampen theorem.

General Constructions

Example 2: Universal Properties

- Universal properties provide general characterizations that introduce unification.
- The categorical product unifies the Cartesian product of sets with conjunction.
  - In **Set**, the categorical product is the Cartesian product.
  - In **Prop**, the categorical product is conjunction.
- The exponential object unifies exponentiation of sets with implication.
  - In **Set**, the exponential object is given by the exponentiation of sets.
  - In **Prop**, the exponential object is given by implication.

General Results

The general constructions given by universal properties often yield general results. For instance, from the two universal properties previously mentioned we get the following result:

Theorem

The categorical product and exponentiation form an adjunction.

This general result relates the general constructions and can be instantiated within specific categories.

General Results

From this general result we get two important corollaries:

Corollary

For propositions α, β, γ,

$$((\alpha \wedge \beta) \to \gamma) \iff (\alpha \to (\beta \to \gamma)).$$

Corollary

For sets A, B, C,

$$\{\, f \mid f : A \times B \to C \,\} \cong \left\{\, f \;\middle|\; f : A \to C^B \,\right\}.$$

Benefits of Generality

The generality of category theory is beneficial because of the unification and new perspectives it introduces. It is through the general constructions and results of category theory that we are able to equate seemingly unrelated constructions such as conjunction and Cartesian product.

Analogy

Category theory provides an excellent framework for developing mathematical analogies. The constructions fundamental to category theory often provide analogies.

Algebraic Invariants

Algebraic invariants allow us to study geometric or topological properties in algebraic terms. They often correspond to a functor or a derived functor.

Example The fundamental group is a functor from the category of based topological spaces to the category of groups.

- This functor points to analogy between the homotopic properties of topological spaces and group theoretic properties — homotopic structure is analogous to group theoretic structure.
- The move from topological spaces to groups introduces a new perspective.

Categorical Equivalence

An equivalence of categories tells us that two categories have the same sophisticated structure. So for each category, we gain a new perspective from which to study it.

Example The category of Boolean algebras is equivalent to the dual of the category of Stone spaces. This equivalence gives us Stone duality for Boolean algebras.

Benefits of Analogy

The analogies of category theory allow us to not only relate one area of mathematics to another, but to reason about one area of mathematics in terms of another. So, similarly to generalizations, analogies provide new perspectives, and these new perspectives better our ability to reason about these areas of mathematics.

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Explanation

- The sophisticated structure of category theory is crucial for producing these generalizations and analogies.
  - It is the emphasis on extrinsic relations that allows for category theory to relate many areas of mathematics,
  - and it is the abstraction away from set theoretic construction that introduces new perspectives.
- I want an account of mathematical explanation that accounts for the role of sophisticated structure in producing explanations.

Explanation as Familiarity

Familiarity To be familiar with some topic or subject matter is to have sufficient knowledge of that topic or subject matter and to be able to reason with that knowledge to draw insightful conclusions.

Explanation A mathematical explanation increases our familiarity with the relevant mathematics.

Sophisticated structure increases our familiarity by relating unfamiliar mathematics to familiar mathematics. So, this characterization of explanation accounts for the role of sophisticated structure.

Benefits

- Allows for a more comprehensive account of mathematical explanation. Explanations can come in the form of proofs, definitions, or informal methods.
- We can also cash out understanding in terms of familiarity to create a clear connection between understanding and explanation.

Conclusion

- Category theory is able to provide explanations from the generalizations and analogies it produces.
- The role that sophisticated structure plays informs a characterization of mathematical explanation.
- More generally, I think the case of category theory and its structural methods can shed light on how we come to know mathematics and how mathematics progresses and develops. So, there is more work to be done in the way of philosophically analyzing the success of category theory.

Thanks for listening!