Lifting Problems in a Grothendieck Fibration

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Definition (Quillen)

Suppose we are given a family of maps $M := (m_i \colon A_i \to B_i)_{i \in I}$ indexed by a set I and a map $f \colon X \to Y$. We say M has the *left lifting property against* f and f has the *right lifting property against* M if for every $i \in I$ and every lifting problem of m_i against f (i.e. every commutative square with m_i on the left and f on the right),

$$\begin{array}{ccc} A_i & \xrightarrow{\ p\ } & X \\ {\scriptstyle m_i}\downarrow & & \downarrow{\scriptstyle f} \\ B_i & \xrightarrow[\ q\ ]{} & Y \end{array}$$

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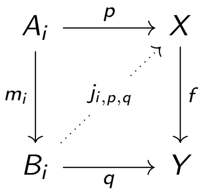

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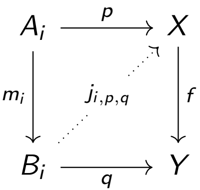

Definition (Garner)

Let $\mathcal{I}$ be a small category and let $M\colon \mathcal{I} \to \mathbb{C}^{\to}$ be a functor. Let $f\colon X \to Y$. We say f has the right lifting property against M if for every object i of $\mathcal{I}$, and every lifting problem of Mi against f

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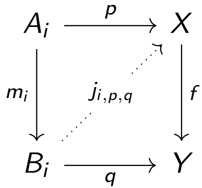

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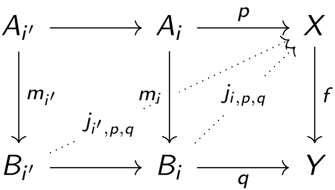

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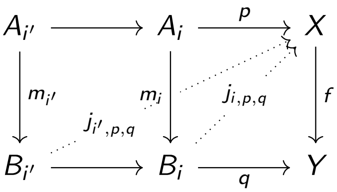

and for every morphism $s\colon i' \to i$ in $\mathcal{I}$, the fillers make a commutative triangle.

Theorem (Garner's Small Object Argument)

Suppose that $\mathbb{C}$ satisfies the following conditions.

1. $\mathbb{C}$ *is locally small.*
2. $\mathbb{C}$ *is cocomplete.*
3. *For every $X \in \mathbb{C}$, there is a regular ordinal α such that $\mathbb{C}(X, -) \colon \mathbb{C} \to \mathsf{Set}$ preserves α-filtered colimits.*

Suppose we are given a small category $\mathcal{I}$ and a diagram $\mathcal{I} \to \mathbb{C}^{\to}$. Then there is a canonical algebraic weak factorisation system (L, R) such that R-algebra structures are precisely such natural choices of lifts. We say (L, R) is cofibrantly generated by M.

Definition (Orton, Pitts)

Suppose that $\mathbb{C}$ is an elementary topos, that $\delta_0, \delta_1 \colon 1 \rightrightarrows \mathbb{I}$ is an interval object and that Φ is a subobject of the subobject classifier. A map $f \colon X \to Y$ is a *Kan fibration* if the following holds in the internal language of $\mathbb{C}$:
For every $\varphi : \Phi$ and $i = 0, 1$, f has the right lifting property against $|\varphi| \hat{\times} \delta_i$, where $|\varphi| \colon \cdot \rightarrowtail 1$ is the proposition classified by φ.

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1. Kan fibration is defined in the internal logic of the topos, without needing an external notion of set.
2. Elementary toposes don't need to be cocomplete. Important examples don't even have colimits of countable sequences (realizability toposes).
3. Constructions carried out in the internal language of a topos are "automatically stable under pullback."

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We will also answer the questions:

1. Is there a version of the small object argument for the internal logic of a topos?
2. If so, in what sense is it stable under pullback?

Definition

Let I be an object of $\mathbb{B}$. A *family of maps indexed by I* is a map $m\colon A \to B$ in $\mathbb{E}_I$.

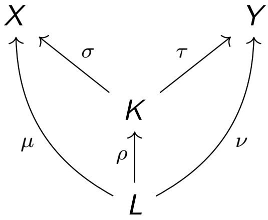
Definition

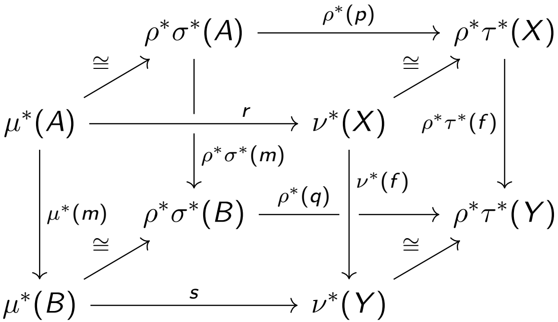
Suppose we are given a families of maps $m\colon A \to B$ over $I \in \mathbb{B}$ and $f\colon X \to Y$ over $J \in \mathbb{B}$. A *family of lifting problems* from m to f consists of an object $K \in \mathbb{B}$, maps $\sigma\colon K \to I$ and $\tau\colon K \to J$ together with a lifting problem of $\sigma^*(m)$ against $\tau^*(f)$ in $\mathbb{E}_K$:

$$\begin{array}{ccc} \sigma^*(A) & \xrightarrow{p} & \tau^*(X) \\ {\scriptstyle \sigma^*(m)}\downarrow & & \downarrow{\scriptstyle \tau^*(f)} \\ \sigma^*(B) & \xrightarrow{q} & \tau^*(Y) \end{array}$$

Definition

A family of lifting problems (K, σ, τ, p, q) is *universal* if every other family of lifting problems factors through it uniquely. That is, for every other family of lifting problems (L, μ, ν, r, s), there is a unique map $\rho\colon L \to K$ making the following diagrams commute.





We will use Bénabou's notion of *locally small* Grothendieck fibration.

Theorem

Suppose that $p\colon \mathbb{E} \to \mathbb{B}$ *is a locally small fibration, and* $\mathbb{B}$ *is finitely complete. Then all universal lifting problems exist.*

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Theorem

Suppose that the universal lifting problem from m to f exists. Then solutions to the universal lifting problem correspond precisely to a coherent choice of solution for every family of lifting problems from m to f.

Definition

We say a family of maps m over I has the *fibred left lifting property* against a family of maps f over J and f has the *fibred right lifting property against* m if the universal lifting problem has a filler.

Example

We consider a category indexed fibration $\mathsf{Fam}_{\mathsf{Cat}}(\mathbb{C}) \to \mathsf{Cat}$ for a locally small category $\mathbb{C}$.

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A family of maps over a small category $\mathcal{I}$ is precisely a map in the functor category $\mathbb{C}^{\mathcal{I}}$. However, $(\mathbb{C}^{\mathcal{I}})^{\to} \cong \mathbb{C}^{\to \times \mathcal{I}} \cong (\mathbb{C}^{\to})^{\mathcal{I}}$. So this is the same as a functor $M \colon \mathcal{I} \to \mathbb{C}^{\to}$.

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f has the fibred right lifting property against M if and only if we have a choice of fillers satisfying Garner's naturality condition.

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A family of maps over $I \in \mathbb{C}$ is a map in the slice category $\mathbb{C}/I$, that is, a map m in a commutative triangle below.

$$\begin{array}{ccc} A & \xrightarrow{\quad m \quad} & B \\ & \searrow \quad \swarrow & \\ & I & \end{array}$$

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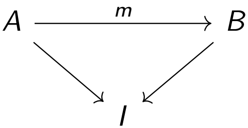

The universal lifting problem is computed as follows. The indexing object K is defined in the internal logic as $K := \sum_{i:I} X^{A_i} \times_{Y^{A_i}} Y^{B_i}$. The horizontal maps in the universal lifting problem are given by evaluation.

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We can view m as a map over 1. Then f has the fibred right lifting property against m if and only if for every object Z of $\mathbb{C}$, f has the (ordinary) right lifting property against $m \times Z$.

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This example can also be understood using enriched lifting problems.

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We can view m as a map into the terminal object of the slice category $\mathbb{C}/B$. f has the fibred right lifting property against m if and only if for every map $g\colon B' \to B$ it has the (ordinary) right lifting property against $g^*(f)\colon g^*(A) \to B'$.

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This example does not appear to be possible to state using enriched lifting problems.

Let $p\colon \mathbb{E} \to \mathbb{B}$ be a locally small bifibration, and assume $\mathbb{B}$ has finite limits.
Suppose we are given a family of maps m over $I \in \mathbb{B}$.
In general we can do the following.

1. Show that for any f the universal family of lifting problems from m to f exists.

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4. Show that the cofibrantly generated awfs exists if and only if we can find a choice of initial algebras for certain pointed endofunctors.

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For codomain fibrations we can use one of the results below.

Theorem (S)

Let $\mathbb{C}$ be a finitely cocomplete, locally cartesian closed category with disjoint sums and W-types.
Suppose that m is a map in a slice category $\mathbb{C}/I$.
If any one of the conditions below holds, then the awfs cofibrantly generated by m exists.

1. *$\mathbb{C}$ has exact quotients and satisfies WISC (a weak choice axiom). This includes all Grothendieck toposes and realizability toposes (as long as WISC holds in the meta theory).*
2. *$\mathbb{C}$ is a category of internal presheaves and m is a locally decidable monomorphism*
3. *$\mathbb{C}$ is boolean*

We can view cofibrantly generated awfs's as monads R over a composition of two Grothendieck fibrations:

$$
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Definition

We say R is *fibred* if it preserves cartesian maps over the Grothendieck fibration $p \circ \mathsf{cod}$.

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We say R is *strongly fibred* if it preserves cartesian maps over the Grothendieck fibration cod.

We can show R is fibred under mild conditions (p is complete and satisfies Beck-Chevalley). In particular, we have,

Theorem

Let $\mathbb{C}$ be a topos that satisfies the Orton-Pitts axioms and WISC. We can define an awfs of trivial cofibrations and Kan fibrations, and this awfs is fibred, but not strongly fibered.

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Usually awfs's will not be strongly fibred, but there is an important exception.

Theorem

Let $\mathbb{C}$ be locally cartesian closed category. We work over the codomain fibration $\mathrm{cod}\colon \mathbb{C}^{\to} \to \mathbb{C}$. Suppose that $m\colon A \to 1_B$ is a map into the terminal object of the slice category $\mathbb{C}/B$. If the awfs cofibrantly generated by m exists, then it is strongly fibred.

In particular the cofibration-trivial fibration awfs in an Orton-Pitts category is stongly fibred.

See these papers for more details:

Swan, *Lifting problems in Grothendieck fibrations*, arXiv:1802:06718

Swan, *W-types with reductions and the small object argument*, arXiv:1802:07588

Some open problems:

1. Can BCH cubical sets be better understood using a suitable Grothendieck fibration?
2. What is the homotopical structure of the Kleene-Vesley topos? How does it compare to topological spaces?

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