Globular PROs as Cartesian-Duoidal Enriched Monoidal Categories

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Motivation

- What would a weak 2-quandle be?

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- In *Higher Operads, Higher Categories* (2004) Tom Leinster proposed a definition of weak ω-categories as algebras for a specific monad.
  - I wanted to show that weak ω-quandles could be expressed as algebras for a similar type of monad

Consequently: I found a way, using globular PROs, to find weak ω versions of any algebraic theory.

PROs

Definition

A **PRO** P is a strict monoidal category whose object set is isomorphic to $\mathbb{N}$ and whose monoidal product $+ : P \times P \to P$ is addition on objects.

- The homset $P(n, m)$ are operations of arity n and coarity m

Algebras over a PRO

Definition

An **algebra for a PRO** P in **Set** is a strict monoidal functor $f : P \to Taut(A)$.

The Tautological (Endomorphism) PRO $Taut(A)$ on a Set A

- Objects are all cartesian powers of A
- $Taut(A)(n, m) = \mathbf{Set}(A^n, A^m)$
- Composition is function composition in **Set**
- Monoidal product is induced by the product structure on **Set**, which is 'addition on objects' since $A^n \times A^m \cong A^{n+m}$

Generalization

Question: How can we combine the globular pasting encoded by Leinster's globular operads and the cartesian algebraic structure encoded by classical PROs into a single object?
Answer: We can enrich over the category over which operads are built!

Globular Sets

Definition

A *globular set* is a contravariant functor $G : \mathbb{G} \to \mathbf{Set}$. The category **Glob** of globular sets is the category of presheaves on $\mathbb{G}$.

Globular Sets

A globular set $G = (\{G_n\}_{n\in\mathbb{N}}, \{s_G^n\}, \{t_G^n\})$ consists of a family of sets $\{G_n\}_{n\in\mathbb{N}}$ together with source and target maps $s_G = \{s_G^n : G_n \to G_{n-1}\}$ and $t_G = \{t_G^n : G_n \to G_{n-1}\}$ subject to the relations $s_G^n \circ s_G^{n+1} = s_G^n \circ t_G^{n+1}$ and $t_G^n \circ s_G^{n+1} = t_G^n \circ t_G^{n+1}$ in each dimension $n \in \mathbb{N}$.

The Free Strict ω-category Monad $\mathcal{T}$: **Glob** $\rightarrow$ **Glob**

Definition

The monad $\mathcal{T}$: **Glob** $\rightarrow$ **Glob** takes a globular set X and returns the underlying globular set of the free strict ω-category generated by X. It extends globular set homomorphisms in the canonical way.

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- Consider the globular set $\mathbf{1}$ with exactly one cell in every dimension.
  - We may think of the elements of $\mathcal{T}(\mathbf{1})$ as all possible unlabeled globular pasting diagrams.

The Free Strict ω-category Monad $\mathcal{T} : \mathbf{Glob} \to \mathbf{Glob}$

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Definition

A **collection** is a globular set X equipped with a globular set homomorphism $x : X \to \mathcal{T}(\mathbf{1})$ called the arity map.

- The category **Col** is the slice category $\mathbf{Glob}/\mathcal{T}(\mathbf{1})$
  - This replaces numeric arities with 'arity shapes'!

Monoidal Structure in **Col**

Definition

Let $x : X \to \mathcal{T}(\mathbf{1})$ and $y : Y \to \mathcal{T}(\mathbf{1})$ be a pair of collections. Their composition tensor product $x \square y : X \square Y \to \mathcal{T}(\mathbf{1})$ is defined by the diagram:

$$
\begin{array}{ccccccc}
X \square Y & \longrightarrow & \mathcal{T}(Y) & \xrightarrow{\mathcal{T}(y)} & \mathcal{T}^2(\mathbf{1}) & \xrightarrow{\mu_1} & \mathcal{T}(\mathbf{1}) \\
\downarrow & \lrcorner & \downarrow{\scriptstyle \mathcal{T}(!_Y)} & & & & \\
X & \xrightarrow{x} & \mathcal{T}(\mathbf{1}) & & & &
\end{array}
$$

where $!_Y : Y \to \mathbf{1}$ is the unique map from Y to the terminal globular set. The underlying globular set $X \square Y$ is the pullback of x and $\mathcal{T}(!_Y)$ with the arity globular set map $x \square y$ defined to be the composition along the top row.

Two Monoidal Structures in **Col**

Composition Tensor Product

- The monoidal unit for $\square$ is the inclusion of generators collection $I : \mathbf{1} \hookrightarrow \mathcal{T}(\mathbf{1})$
- A **globular operad** is a monoid in **Col** with respect to the tensor product $\square$

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Cartesian Product

As a slice category, **Col** has a cartesian product defined as the pullback (in **Glob**) of two collection arity maps

- A product consists of ordered pairs of n-cells, all of which have the same 'arity shape' in $\mathcal{T}(\mathbf{1})$
- The monoidal unit for $\times$ is the terminal collection $\mathbb{1} : \mathcal{T}(\mathbf{1}) \to \mathcal{T}(\mathbf{1})$

These two monoidal structures satisfy 'nice' compatibility axioms:

Duoidal Category

Definition - Batanin, Markl (2012)

A **duoidal category** $(\mathcal{D}, \otimes, I, \odot, U, \delta, \phi, \theta, \boxplus)$ consists of a category $\mathcal{D}$, a pair of 2-variable functors $\otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ and $\odot : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$, a pair of unit objects I and U, three morphism $\delta : I \to I \odot I$, $\phi : U \otimes U \to U$, and $\theta : I \to U$ in $\mathcal{D}$, and a lax middle-four interchange natural transformation

$$\boxplus : \otimes(\odot(-,-), \odot(-,-)) \Rightarrow \odot(\otimes(-,-), \otimes(-,-))$$

$$\boxplus_{A,B,C,D} : [A \odot B] \otimes [C \odot D] \to [A \otimes C] \odot [B \otimes D]$$

such that

- $(\mathcal{D}, \otimes, I)$ and $(\mathcal{D}, \odot, U)$ are both monoidal structures on $\mathcal{D}$
- U is a monoid object in $(\mathcal{D}, \otimes, I)$ and I is a comonoid object in $(\mathcal{D}, \odot, U)$

They are pseudomonoid in the category of monoidal categories and lax-monoidal functors.

Enriched $\mathcal{D}$-categories

Definition

A **category enriched over a duoidal category** $(\mathcal{D}, \otimes, I, \odot, U, \delta, \phi, \theta, \boxplus)$, or simply a $\mathcal{D}$-category, is an enriched category with respect to the monoidal structure $(\mathcal{D}, \otimes, I)$.

- So why is this interesting?

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- So why is this interesting?
- The second monoidal structure from our duoidal category $\mathcal{D}$ induces a monoidal structure on $\mathcal{D}\mathbf{Cat}$, the category of categories enriched over $\mathcal{D}$.

Monoidal Structure on $\mathcal{D}\mathbf{Cat}$

Definition

The tensor product

$$\oplus : \mathcal{D}\mathbf{Cat} \times \mathcal{D}\mathbf{Cat} \to \mathcal{D}\mathbf{Cat}$$

of $\mathcal{D}$-categories $\mathcal{E}$ and $\mathcal{F}$ is given as the Cartesian product on objects and for $A, B \in Obj(\mathcal{E})$ and $X, Y \in Obj(\mathcal{F})$ we have

$$\mathcal{E} \oplus \mathcal{F}((A, X), (B, Y)) := \mathcal{E}(A, B) \odot \mathcal{F}(X, Y)$$

as the hom-objects in $\mathcal{E} \oplus \mathcal{F}$, where $\odot$ is the second monoidal structure on $\mathcal{D}$.

- The unit $\mathcal{D}$-category $\mathbf{1}_{\oplus}$ for $\oplus$ consists of a single object $*$ and a single hom-object $\mathbf{1}_{\oplus}(*, *)$ given by the monoidal unit U for the second monoidal structure in $\mathcal{D}$.

Cartesian-Duoidal Enriched Monoidal Categories

Definition

A **monoidal $\mathcal{D}$-category** is a pseudomonoid in the category $\mathcal{D}$**Cat** of categories duoidally enriched over the duoidal category $\mathcal{D}$.

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A **cartesian-duoidal category** is a duoidal category $(\mathcal{D}, \otimes, I, \times, \mathbb{1}, \delta, \phi, \theta, \boxplus)$ such that the second monoidal structure $(\mathcal{D}, \times, \mathbb{1})$ is cartesian.

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Definition

A **cartesian-duoidal enriched monoidal category** is a pseudomonoid in $(\mathcal{C}\mathbf{Cat}, \oplus, \mathbf{1}_{\oplus})$, the monoidal category of categories duoidally enriched over a cartesian-duoidal category $\mathcal{C}$.

Globular PROs

Definition

A **globular PRO** is a cartesian-duoidal enriched monoidal category $(\mathcal{P}, +, O)$ enriched over the cartesian-duoidal category **Col** such that

- The object set of $\mathcal{P}$ is $\mathbb{N}$
- The bifunctor $+ : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is addition on objects

Definition

An **algebra for a globular PRO** $\mathcal{P}$ is a strict monoidal **Col**functor $F : \mathcal{P} \to GTaut(A)$ to the globular tautological PRO on a globular set A.

PRO Globularization

Given an ordinary PRO P, we can construct a globular PRO $\mathcal{P}$ whose algebras are exactly the strict ω-categories which are algebras for P whose operations in P are given by strict ω-functors.

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Definition

Let P be any ordinary set PRO and consider the functor $G_P : P \to \mathcal{P}$ which maps P to its globularization $\mathcal{P}$.

$$n \mapsto n$$

$$P(n,m) \mapsto \mathcal{P}(n,m) := P(n,m) \cdot \mathbb{1} = \coprod_{P(n,m)} \mathcal{T}(\mathbf{1})$$

Note: We lift via the cartesian unit $\mathbb{1} : \mathcal{T}(\mathbf{1}) \to \mathcal{T}(\mathbf{1})$ rather than the globular composition unit $I : \mathbf{1} \hookrightarrow \mathcal{T}(\mathbf{1})$ (which is a sub-object) so that the pasting composition described by $\mathcal{T}(\mathbf{1})$ is combined with the operations lifted from P.

PRO Globularization Induced Composition

For all $n, m, p \in \mathbb{N}$ the hom-object $\mathcal{P}(m,p) \square \mathcal{P}(n,m)$ can be written

$$\mathcal{P}(m,p) \square \mathcal{P}(n,m) = \left(\coprod_{P(m,p)} \mathcal{T}(\mathbf{1}) \right) \square \left(\coprod_{P(n,m)} \mathcal{T}(\mathbf{1}) \right) =$$

$$\coprod_{P(m,p)} \left(\mathcal{T}(\mathbf{1}) \square \left(\coprod_{P(n,m)} \mathcal{T}(\mathbf{1}) \right) \right) = \coprod_{P(m,p)} \left(\coprod_{P(n,m)} (\mathcal{T}(\mathbf{1}) \square \mathcal{T}(\mathbf{1})) \right)$$

$$\cong \coprod_{P(m,p) \times P(n,m)} (\mathcal{T}(\mathbf{1}) \square \mathcal{T}(\mathbf{1}))$$

Which allows us to define

$$\circ^{\mathcal{P}}_{n,m,p} := \circ^{P}_{n,m,p} \cdot \phi : (P(m,p) \times P(n,m)) \cdot (\mathbb{1} \square \mathbb{1}) \to P(n,p) \cdot \mathbb{1}$$

The cartesian unit $\mathbb{1}$ is a monoid object with respect to $\square$ with multiplication $\phi : \mathcal{T}(\mathbf{1}) \square \mathcal{T}(\mathbf{1}) \to \mathcal{T}(\mathbf{1})$

PRO Globularization Induced Addition

Similarly, for all $n, m, l, k \in \mathbb{N}$, we can rewrite

$$\mathcal{P}(n,m) \times \mathcal{P}(l,k) = \left(\coprod_{P(n,m)} \mathcal{T}(\mathbf{1}) \right) \times \left(\coprod_{P(l,k)} \mathcal{T}(\mathbf{1}) \right) =$$

$$\coprod_{P(n,m)} \left(\mathcal{T}(\mathbf{1}) \times \left(\coprod_{P(l,k)} \mathcal{T}(\mathbf{1}) \right) \right) = \coprod_{P(n,m)} \left(\coprod_{P(l,k)} (\mathcal{T}(\mathbf{1}) \times \mathcal{T}(\mathbf{1})) \right)$$

$$\cong \coprod_{P(n,m) \times P(l,k)} (\mathcal{T}(\mathbf{1}) \times \mathcal{T}(\mathbf{1}))$$

Allowing us to define

$$+^{\mathcal{P}}_{n,m,l,k} := +^{P}_{n,m,l,k} \cdot \Phi : (P(n,m) \times P(l,k)) \cdot (\mathbb{1} \times \mathbb{1}) \to P(n+l, m+k) \cdot \mathbb{1}$$

$$\Phi : \mathbb{1} \times \mathbb{1} \xrightarrow{\cong} \mathbb{1}$$

Strict ω-structures

How to Strict ω-ify Your Favorite Agebraic Theory

- Consider your favorite algebraic theory.
- Find a description of the PRO for that theory.
- Globularize the ordinary PRO.
- Algebras for the globularization are strict ω-categorifications of the original theory.

So Why Have We Really Done This?

Leinster's contractions naturally adapt to this situation.

Definition

A **contraction structure** on a map $f : X \to Y$ of globular sets is a choice of lifts for every cell $\gamma \in Y$ whose boundary is the image of a pair of parallel cells in X.

Definition

A **contraction structure on a globular PRO** $\mathcal{P}$ is a map of globular PROs $F : \mathcal{P} \to \mathcal{P}'$ such that each component $F_{n,m} : P(n, m) \to P'(n, m)$ of its underlying **Col**-functor comes equipped with a specified contraction.

Leinster's Free Contraction Construction

In appendix G of Leinster's *Higher Operads Higher Categories* he describes a functorial construction for expanding a generic globular set map to get one which has a natural induced contraction.

- This construction naturally extends to maps of $\mathbb{N}\mathbf{Col}$-graphs by applying it at each hom-object.

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In appendix G of Leinster's *Higher Operads Higher Categories* he describes a functorial construction for expanding a generic globular set map to get one which has a natural induced contraction.

- This construction naturally extends to maps of $\mathbb{N}$**Col**-graphs by applying it at each hom-object.

- For any globular PRO $\mathcal{P}$ this gives a functor $C_{\mathcal{P}} : {}^{\mathbb{N}\mathbf{ColGraph}}/_{\mathcal{U}(\mathcal{P})} \to \mathbf{Cont}\left({}^{\mathbb{N}\mathbf{ColGraph}}/_{\mathcal{U}(\mathcal{P})}\right)$
- The right adjoint that simply forgets the contraction structure

$$h_{\mathcal{P}} : \mathbf{Cont}\left({}^{\mathbb{N}\mathbf{ColGraph}}/_{\mathcal{U}(\mathcal{P})}\right) \to {}^{\mathbb{N}\mathbf{ColGraph}}/_{\mathcal{U}(\mathcal{P})}$$

Two Key Structures on Globular PROs

The free monoid and free path category functors each have obvious right adjoints

- $\mathcal{W}$: **Mon**$\mathbb{N}$**ColGraph** $\to$ $\mathbb{N}$**ColGraph**
- $\mathcal{U}$: $\mathbb{N}$**ColCat** $\to$ $\mathbb{N}$**ColGraph**

Lemma

Given a globular PRO $\mathcal{P}$, the induced functors on slice categories

$$\overline{\mathcal{W}}_{\mathcal{P}} : \mathbf{Mon}\mathbb{N}\mathbf{ColGraph}/\mathcal{U}(\mathcal{P}) \longrightarrow \mathbb{N}\mathbf{ColGraph}/\mathcal{U}(\mathcal{P})$$

and

$$\overline{\mathcal{U}}_{\mathcal{P}} : \mathbb{N}\mathbf{ColCat}/\mathcal{P} \longrightarrow \mathbb{N}\mathbf{ColGraph}/\mathcal{U}(\mathcal{P})$$

are monadic over $\mathbb{N}\mathbf{ColGraph}/\mathcal{U}(\mathcal{P})$.

The Free Globular PRO with Contraction Monad

We have three monadic functors:

- $h_P : \mathbf{Cont}\left({}^{\mathbb{N}}\mathbf{ColGraph}/\mathcal{U}(\mathcal{P})\right) \longrightarrow {}^{\mathbb{N}}\mathbf{ColGraph}/\mathcal{U}(\mathcal{P})$
- $\overline{\mathcal{W}}_{\mathcal{P}} : \mathbf{Mon}{}^{\mathbb{N}}\mathbf{ColGraph}/\mathcal{U}(\mathcal{P}) \longrightarrow {}^{\mathbb{N}}\mathbf{ColGraph}/\mathcal{U}(\mathcal{P})$
- $\overline{\mathcal{U}}_{\mathcal{P}} : {}^{\mathbb{N}}\mathbf{ColCat}/\mathcal{P} \longrightarrow {}^{\mathbb{N}}\mathbf{ColGraph}/\mathcal{U}(\mathcal{P})$

The pullback of which can be shown, using Kelly's theorem regarding algebraic colimits, to be monadic. We shall denote the monad constructed by this pullback

$$\mathfrak{G}_P : {}^{\mathbb{N}}\mathbf{ColGraph}/\mathcal{U}(\mathcal{P}) \circlearrowleft$$

It's algebras are globular PROs with contraction over $\mathcal{P}$.

The initial algebra for $\mathfrak{G}_P$ is the globular PRO whose algebras are the fully weakened ω-categorifications for the theory described by the classic PRO P!

Thank You

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