Modeling Language and Translations, Categorically

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How might we understand language, mathematically?

syntax **semantics**

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syntax

semantics

The Yoneda lemma for linguistics:

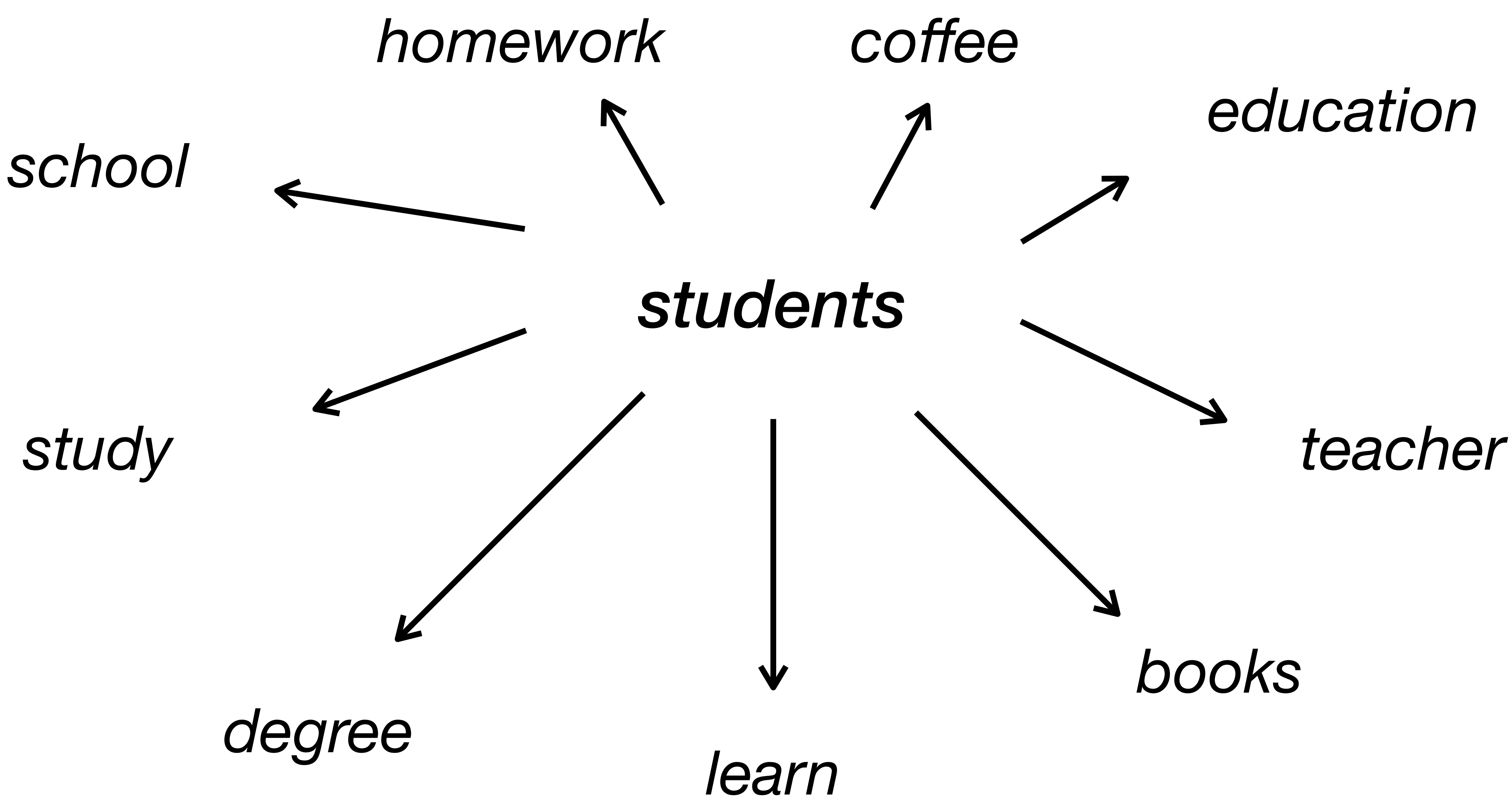
“You shall know a word by the company it keeps.”

– John Firth, 1957

(a.k.a. the distributional hypothesis)

How might we understand language, mathematically?

syntax **semantics**



How might we understand language, mathematically?

syntax

semantics

$$\text{students} \quad \mapsto \begin{bmatrix} 8 \\ 5 \\ 0 \end{bmatrix} = \mathbf{students} \in \mathbb{R}^3$$

where $\mathbb{R}^3$ is generated by

$$\mathbf{school} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{books} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{butterfly} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

How might we understand language, mathematically?

syntax

semantics

$$\|$$

$$(\mathsf{FVect}, \otimes, \mathbb{R})$$

We choose our semantics category to be the category of **finite-dimensional real vector spaces**. It's a symmetric monoidal category. It's also compact closed (every object has a dual).

- monoidal product = tensor product
- monoidal unit = ground field

How might we understand language, mathematically?

syntax semantics

cordially
adverb

in
preposition

students $\longrightarrow$ ***learn***
noun verb

purple
adjective

sweater
noun

How might we understand language, mathematically?

syntax

semantics

$$\text{syntax} \; = \; (\mathsf{G}, \cdot, 1)$$

We choose our syntax category to be the **free compact closed category** $(\mathsf{G}, \cdot, 1)$ **on a finite set of grammar types**. Example: if the set is $\{n, s\}$:

$n =$ noun

$s =$ sentence

$n^r s =$ intransitive verb

$\vdots$

$$\begin{array}{cc} \textit{students} & \textit{learn} \\ n & n^r s \end{array}$$

$$\downarrow$$

$$\begin{array}{c} \textit{students learn} \\ s \end{array}$$

Definition: Let G be a free compact closed category on a finite set of grammar types. A distributional categorical language model—or **language model**, for short—is a strong monoidal functor

$$(\mathsf{G}, \cdot, 1) \xrightarrow{F} (\mathsf{FVect}, \otimes, \mathbb{R})$$

Every grammar type g in G corresponds to a vector space Fg and every grammar reduction $r\colon g \to h$ gives rise to a linear transformation $Fr\colon Fg \to Fh$.

(pause)

A little background:

- In 2010, Coecke et. al. model language via the Cartesian product G × FVect.
  - Coecke, B., Sadrzadeh M., and Clark, S. “Mathematical foundations for a compositional distributional model of meaning.” *arXiv:1003.4394*
- In 2014, Kartsaklis et. al. model language via a strong monoidal functor $\mathsf{G} \to \mathsf{FVect}_V$.
  - Kartsaklis, D., Sadrzadeh, M., Pulman, S. and Coecke, B. “Reasoning about meaning in natural language with compact closed categories and Frobenius algebras.” *Logic and Algebraic Structures in Quantum Computing and Information,* p. 199; *arXiv:1401.5980*

(unpause)

Definition: Let G be a category that is freely monoidal on a finite set of grammar types. A distributional categorical language model—or **language model**, for short—is a strong monoidal functor

$$(\mathsf{G}, \cdot, 1) \xrightarrow{F} (\mathsf{FVect}, \otimes, \mathbb{R})$$

To pair grammar types with meaning vectors, e.g.

$$n \rightsquigarrow \begin{bmatrix} 8 & 5 & 0 \end{bmatrix}^\top = \mathbf{students}^\top$$

we wish to use the Grothendieck construction.

But… F is not Cat-valued!

There is a functor

$$\mathsf{FVect} \xrightarrow{C} \mathsf{Cat}$$

$$\begin{tikzcd} V \arrow[d, "f"'] & & CV \arrow[d, "Cf"] \\ W & \longmapsto & CW \end{tikzcd}$$

where CV is the category whose objects are vectors $\mathbf{v} \in V$ and there is a unique morphism $\mathbf{v} \to \mathbf{v}'$ labeled by the Euclidean distance $d(\mathbf{v}, \mathbf{v}')$. For any linear map $f\colon V \to W$ define $Cf\colon CV \to CW$ to be the functor that agrees with f on objects and is given by $d(\mathbf{v}, \mathbf{v}') \mapsto d(f\mathbf{v}, f\mathbf{v}')$ on morphisms.

Definition: Let F be a language model and let C be as before.

$$\mathsf{G} \xrightarrow{F} \mathsf{FVect} \xrightarrow{C} \mathsf{Cat}$$

The **product space representation** of F with respect to C, denoted $\int_C F := \int CF$, is the Grothendieck construction of CF. Explicitly, it is a category with

objects:

$$(g, \mathbf{w})$$

where

$$\mathbf{w} \in CFg$$

morphisms:

$$(g, \mathbf{w}) \xrightarrow{(r,d)} (h, \mathbf{v})$$

where

$$g \xrightarrow{r} h \qquad d := d(CFr\mathbf{w}, \mathbf{v})$$

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$$g \xrightarrow{r} h \qquad d := d(CFr\mathbf{w}, \mathbf{v})$$

Proposition: Let F be a language model. Then $\int_C F$ is a monoidal category.

- On objects, the monoidal product is

$$(g, \mathbf{w}) \otimes (h, \mathbf{v}) = (gh, \Phi_{g,h}(\mathbf{w} \otimes \mathbf{v}))$$

- On morphisms, the monoidal product is

$$(r, d) \otimes (r', d') = (rr', \Phi_{g,h}(d \otimes d'))$$

where $\Phi_{g,h} : CFg \otimes CFg \to CF(gh)$ is the natural isomorphism included in the data of the strong monoidal functor CF.

Next, we want to make sense of the assignment

$$\text{students} \mapsto (n, \textbf{students})$$

Definition: Let F be a language model and let W be the free monoid on a finite set of words, viewed as a discrete category. A **lexicon** is a functor

$$\begin{array}{c} W \\ \big\downarrow{\scriptstyle \ell} \\ \int_C F \end{array}$$

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$$\begin{tikzcd} W \arrow[d, "\ell"] \\ \int_C F \end{tikzcd} \qquad \begin{tikzcd} \text{students} \arrow[d] \\ (n, \mathbf{students}) \end{tikzcd} \qquad \begin{tikzcd} \text{learn} \arrow[d] \\ (n^r s, \mathbf{learn}) \end{tikzcd}$$

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$$\begin{array}{ccc} W & & \text{students learn} \\ \Big\downarrow \ell & & \Big\downarrow \\ \int_C F & & (nn^r s, \mathbf{students} \otimes \mathbf{learn}) \\ & & := \ell(\text{students}) \otimes \ell(\text{learn}) \end{array}$$

Recap

We can model language by

- considering a monoidal functor grammar → vector spaces
- use the Grothedieck construction to pair grammar types with meaning vectors
- view words in a text as a discrete category, and map them to the Grothendieck construction.

But what if we have *two* language models?

$$?$$

English | **Spanish**

"students learn"

$n_E \quad n_E^r s_E$

"estudiantes aprenden"

$n_S \quad n_S^r s_S$

$F\colon \mathsf{G} \longrightarrow \mathsf{FVect}$

$F'\colon \mathsf{G}' \longrightarrow \mathsf{FVect}$

G is free on $\{n_E, s_E\}$

G' is free on $\{n_S, s_S\}$

Definition: A **translation** $T = (j, \alpha)$ from a language model F to a language model F' is a monoidal functor j and a monoidal natural transformation $\alpha\colon F \Rightarrow F'j$.

$$
\begin{tikzcd}
\mathsf{G} \arrow[rr, "F"] \arrow[d, "j"'] \arrow[dr, phantom, "\Downarrow\alpha"] & & \mathsf{FVect} \\
\mathsf{G}' \arrow[urr, "F'"'] & &
\end{tikzcd}
$$

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\end{tikzcd}$$

$$\alpha_{n_E} : F(n_E) \to F'(n_s)$$
$$\alpha_{s_E} : F(s_E) \to F'(s_s)$$
$$\vdots$$

Example

$$j(n_E) = n_S$$
$$j(s_E) = s_S$$
$$F(n_E) = N_E$$
$$F'(n_S) = N_S$$
$$F(s_E) = S_E$$
$$F('s_S) = S_S$$

DisCoCat

is the category with

objects:

language models

$$(\mathsf{G}, \cdot, 1) \xrightarrow{F} (\mathsf{FVect}, \otimes, \mathbb{R})$$

Distributional
Compositional
Categorical
language models

morphisms:

translations

$$\begin{tikzcd}
\mathsf{G} \arrow[r, "F"] \arrow[d, "j"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathsf{FVect} \\
\mathsf{G}' \arrow[ur, "F'"'] &
\end{tikzcd}$$

Proposition. Let $K\colon \mathsf{FVect} \to \mathsf{Cat}$ be a fully faithful functor and let MonCat denote the category of monoidal categories and strong monoidal functors. There is a functor

$$\int_K (-)\colon \mathsf{DisCoCat} \to \mathsf{MonCat}$$

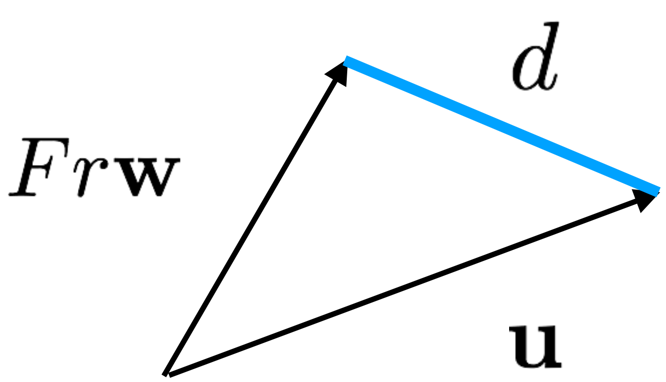
$$\begin{array}{ccc} F & & \int_K F \\ {\scriptstyle T}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle \int_K T} \\ F' & & \int_K F' \end{array}$$

where $\int_K T$ is the strong monoidal functor given by…

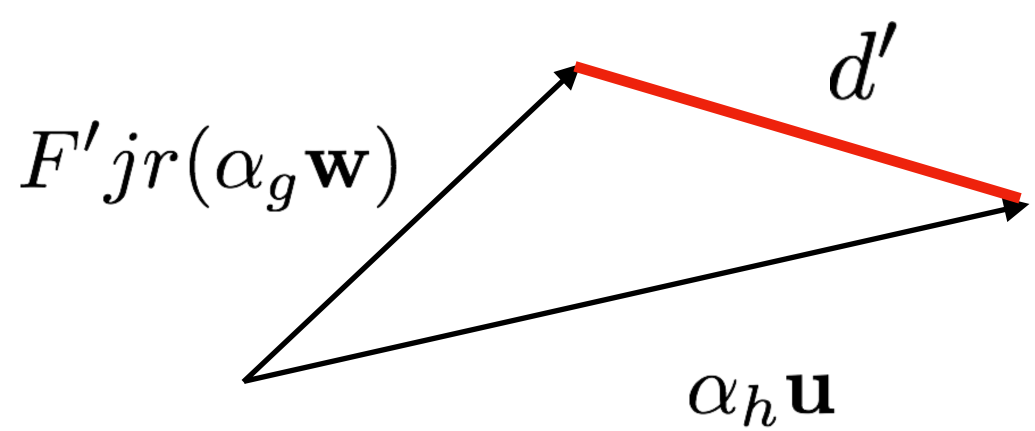
(take K = C)

$$\int_C F \xrightarrow{\quad \int_C T \quad} \int_C F'$$

$$\begin{tikzcd}
(g, \mathbf{w}) \arrow[d, "{(r,d)}"'] \arrow[r, phantom, "\longmapsto"] & (jg, \alpha_g \mathbf{w}) \arrow[d, "{(jr,d')}"] \\
(h, \mathbf{u}) & (jh, \alpha_h \mathbf{u})
\end{tikzcd}$$



in Fh



in $F'jh$

Definition: Let $\ell \colon W \to \int_C F$ and $\ell' \colon W' \to \int_C F'$ be lexicons, and let $T = (j, \alpha)$ be a translation from F to F'. The F-F' **dictionary** with respect to T is the comma category

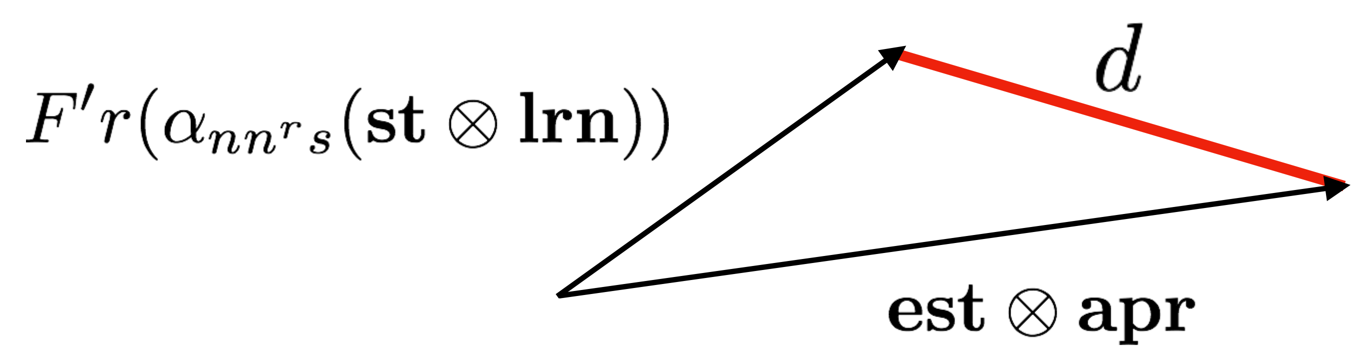
$$\left(\int_C T \circ \ell \right) \downarrow \ell'$$

Concretely, it is the set (discrete category) of triples $(w, (r,d), w')$ where $w \in W$ and $w' \in W'$ and $(r,d) \colon \int_C T(\ell w) \to \ell' w'$ is a morphism in $\int_C F'$.

Example: Suppose W and W' are as below, and we have a translation $T = (j, \alpha)$ between language models F and F' (as before), where α can be determined appropriately.

$$W = \{\text{students}, \text{ learn}, \ldots\}$$
$$W' = \{\text{estudiantes}, \text{ aprenden}, \ldots\}$$

then $(\text{students learn}, (r, d), \text{estudiantes aprenden})$ is an object in the F-F' dictionary, where $j(n_E n_E^r s_E) \xrightarrow{r} n_S n_S^r s_S$ is a reduction in G' and the distance d is determined by first translating $\mathbf{st} \otimes \mathbf{lrn}$ then applying the linear map corresponding to the reduction.



in $F(n_s) \otimes F(n_s) \otimes F(s_S)$

Future Work

1. Adapt the model to handle change in word order: e.g. "red car" vs. "coche rojo." (In this talk, we only considered the fragment of language consisting of nouns/intransitive verbs)

2. Take advantage of string diagram calculus.

3. Used an enriched version (Lawvere metric spaces).

4. Investigate meaning change and negotiated meaning between speakers (language evolution).

Thank you!